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Unit-1 Differential Calculus

* Task-1 Limit, continuity and Differentiability

1 Evaluate $\lim_{x \to -1} \frac{(x+3)(2x-1)}{x^2+3x-2}$

$= \lim_{x \to -1} \frac{(x+3)(2x-1)}{x^2+3x-2}$

$= \frac{(-1+3)(2(-1)-1)}{(-1)^2+3(-1)-2}$

$= \frac{(2)(-2-1)}{1-3-2}$

$= \frac{2(-3)}{-4}$

$= \frac{3}{2}$

2 Evaluate $\lim_{x \to -1} \frac{2x^3-3x^2-3x+2}{3x^3+2x^2-11x-10}$

$= \lim_{x \to -1} \frac{(x+1)(2x^2-5x+2)}{(x+1)(3x^2-x-10)}$

$$= \lim_{x \to -1} \frac{2x^2 - 5x + 2}{3x^2 - x - 10}$$

$$= \frac{2(-1)^2 - 5(-1) + 2}{3(-1)^2 - (-1) - 10}$$

$$= \frac{2 + 5 + 2}{3 + 1 - 10}$$

$$\neq \cancel{\frac{10}{14}}$$

$$= \frac{9}{-6}$$

$$= \frac{-3}{2}$$

3 Evaluate $\lim_{x \to 1/5} \frac{125x^3 - 1}{625x^4 - 1}$

$$= \lim_{x \to 1/5} \frac{(5x)^3 - (1)^3}{(5x)^4 - (1)^4}$$

$$= \lim_{x \to 1/5} \frac{(5x)^3 - (1)^3}{x - 1} \times \frac{x - 1}{(5x)^4 - (1)^4}$$

$= \cancel{\lim_{x \to 1/5}}$

$= \frac{3(1)^{3-1}}{4(1)^{4-1}}$

$= \frac{3}{4}$

4 Discuss the Continuity of

$$F(x) = \frac{5}{x-1}, \quad x \neq 1$$
$$= 1, \quad x = 1$$

$\therefore \lim_{x \to 1} F(x) = \lim_{x \to 1} \frac{5}{x-1}$

$= \lim \frac{5}{1-1}$

$= \infty$

$\therefore \lim_{x \to 1} F(x) \neq F(1)$

$\therefore$ f(x) is not Continuity at $x = 1$

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5 Discuss the Continuity of

$$f(x) = \begin{cases} 2x+3, & x<1 \\ 5, & x=1 \\ 2+3x, & x>1 \end{cases}$$

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

$$\therefore \lim_{x \to 1^+} 2+3x = \lim_{x \to 1^-} 2x+3$$

$$\therefore 2+3(1) = 2(1)+3$$

$$\therefore 5 = 5$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$f(x)$ is Continuity at $x=1$

6 Let $f(x) = \sqrt{2x-1}$ than find $f'(5)$

$$\therefore f'(5) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{10+2h-1} - \sqrt{10-1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{9+2h} - 3}{h} \times \frac{\sqrt{9+2h} + 3}{\sqrt{9+2h} + 3}$$

$$= \lim_{h \to 0} \frac{9+2h+9}{h(\sqrt{9+2h}+3)}$$

$$= \lim_{h \to 0} \frac{2h}{h(\sqrt{9+2h}+3)}$$

$$= \frac{2}{\sqrt{9+2(0)}+3}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

7 If $f(x) = \sin^2 x$ then find $f'(x)$ by definiton.

$$f'(x) = \lim_{h \to 0} \frac{(\sin(x+h)^2) - (\sin x)^2}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h)^2 - \sin^2 x}{h}$$

$$= \lim_{h \to 0} \sin^2 x \cdot \frac{\cos h}{h} (\cos h + 1) +$$

$$2\sin x\cos h\cos x\cdot\frac{\sin h}{h}+$$
$$\cos^2 x\cdot\frac{\sin h}{h}\cdot\sin h$$

$$=\lim_{h\to 0}\ \sin^2(0)(0)((\cos 0+1)+2\sin((\cos 0)$$
$$\cos x(1)+\cos(x)\sin 0$$

$$=2\sin x\cos x$$

8 Evaluate the limit $\lim_{x\to -8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$

$$=\lim_{x\to -8}\frac{(1-x)^{\frac{1}{2}}-3}{2+(x)^{1/3}}$$

$$=\lim_{x\to -8}\frac{\frac{1}{2}(1-x)^{-\frac{1}{2}+1}}{\frac{1}{3}(x)^{-\frac{2}{3}}}$$

$$=\lim_{x\to -8}\frac{\frac{1}{2}\sqrt{1-x}}{\frac{1}{3}(x)^{-2/3}}$$

$$=\frac{1}{2}\frac{\sqrt{1-(-8)}}{(2^3)^{-\frac{2}{3}}}\times 3$$

$$=2$$

* Task 2 : Sandwich Theorem

1 Find $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

We know that,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

By Sandwich Theorem,

$$\therefore -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\therefore -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\therefore \lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$$

$$\therefore 0 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

By Sandwhich Theorem $x^2 \sin\left(\frac{1}{x}\right) = 0$

2 IF $5x \leq f(x) \leq 2x^2 + 2$, $\forall_x \in R$ then Find $\lim_{x \to 2} f(x)$

Given, $5x \leq f(x) \leq 2x^2 + 2$

$$\therefore \lim_{x \to 2} 5x \leq \lim_{x \to 2} f(x) \leq \lim_{x \to 2} 2x^2 + 2$$

$\therefore 5(2) \leq \lim_{x \to 2} f(x) \leq 2(2)^2 + 2$

$\therefore 10 \leq \lim_{x \to 2} f(x) \leq 10$

By Sandwich Theorem, $\lim_{x \to 2} f(x) = 10$.

3 Find $\lim_{x \to 0} g(x)$, if $3 - x^3 \leq g(x) \leq 3 \sec x$ for all $x \in R$.

Given, $3 - x^3 \leq g(x) \leq 3 \sec x$

$\therefore \lim_{x \to 0} 3 - x^3 \leq \lim_{x \to 0} g(x) \leq \lim_{x \to 0} 3 \sec x$

$\therefore 3 \leq \lim_{x \to 0} g(x) \leq 3$

By Sandwich Theorem, $\lim_{x \to 0} g(x) = 3$

4 Evaluate $\lim_{x \to \infty} \frac{\cos^2(2x)}{3-2x}$

We know that, $0 \leq \cos^2 2x \leq 1$

By Sandwich theroum,

$$\therefore 0 \leq \cos^2 2x \leq 1$$

$$\therefore 0 \leq \frac{\cos^2 2x}{3-2x} \leq \frac{1}{3-2x}$$

$$\therefore \lim_{x \to \infty} 0 \leq \lim_{x \to \infty} \frac{\cos^2(2x)}{3-2x} \leq \lim_{x \to \infty} \frac{1}{3-2x}$$

$$\therefore 0 \leq \lim_{x \to \infty} \frac{\cos^2(2x)}{3-2x} \leq 0$$

By Sandwich Theorem, $\lim_{x \to \infty} \frac{\cos^2(2x)}{3-2x} = 0$

5 Prove that $\lim_{x \to 0} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) = 0$

We know that, $0 \leq \sin^2\left(\frac{2\pi}{x}\right) \leq 1$

By Sanwich theorem,

$$\therefore 0 \leq \sin^2\left(\frac{2\pi}{x}\right) \leq 1$$

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$$\therefore 1 \leq 1 + \sin^2\left(\frac{2\pi}{x}\right) \leq 2$$

$$\therefore \sqrt{x} \leq \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) \leq 2\sqrt{x}$$

$$\therefore \lim_{x \to 0} \sqrt{x} \leq \lim_{x \to 0} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) \leq \lim_{x \to 0} 2\sqrt{x}$$

$$\therefore 0 \leq \lim_{x \to 0} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) \leq 0$$

By Sandwich theorem,

$$\lim_{x \to 0} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) = 0$$

* Task 3 : Indeterminate Forms

1 $\lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x}$

Given $l = \lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x}$ $\left(\frac{0}{0}\right)$ form

By L' Hospital,

$\therefore\ l = \lim_{x \to 1} \frac{x^x(1 + \log x) - 1}{1 - \frac{1}{x}}$ $\left(\frac{0}{0}\right)$ form

By L' Hospital,

$\therefore\ l = \lim_{x \to 1} \frac{x^x\left(\frac{1}{x}\right) + (1 + \log x)(1 + \log x)}{+ 1/x^2}$

$\therefore\ l = 1 + 1$

$= 2$

2. $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

given, $l = \lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\left(\frac{0}{0}\right)$ form

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By L' Hospital,

$$l = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \quad \left(\frac{0}{0}\right) \text{ form,}$$

By L' Hospital,

$$\therefore l = \lim_{x \to 0} \frac{\sin x}{6x}$$

$$\therefore l = \lim_{x \to 0} \frac{1}{6} \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$l = \frac{1}{6}$$

3 $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

$$= \frac{\lim_{x \to 0} (1+x)^{\frac{1}{x}} - \lim_{x \to 0} + e}{\lim_{x \to 0} x}$$

$$= \lim_{x \to 0} \frac{e \log (1+x)^{\frac{1}{x}} - e}{x}$$

$$= \lim_{x \to 0} e^{\frac{1}{x} \log(1+x)} \left[-\frac{1}{x^2} \log(1+x) + \frac{1}{x} \cdot \frac{1}{1+x} \right]$$

Imp

$$= e \lim_{x \to 0} \frac{-\log(1+x)(1+x)}{x^2(1+x)}$$

$$= e \lim_{x \to 0} \frac{-1-\log(1+x)+1}{2x+3x^2}$$

$$= e \lim_{x \to 0} \frac{-\frac{1}{1+x}}{2+6x} = \frac{-e}{2}$$

Imp

4 Determine value of a, b, c so that $\lim_{x \to 0} \frac{ae^{-x} - b\cos x + ce^{-x}}{x\sin x} = 2$

given $l = \lim_{x \to 0} \frac{ae^{-x} - b\cos x + ce^{-x}}{x\sin x} = 2$

$$\therefore \lim_{x \to 0} \frac{ae^{-x} - b\cos x + ce^{-x}}{x\sin x} = 2 \quad \left(\frac{0}{0}\right)$$

By L'Hospital,

$$\therefore \lim_{x \to 0} \frac{-ae^{-x} + b\sin x - ce^{-x}}{x\cos x + \sin x} = 2$$

$$\therefore \frac{-ae^{-0} + b\sin 0 - ce^{-0}}{0+0} = 0$$

$$\therefore \cancel{b\sin 0 = 2}$$

$$\therefore \cancel{\boxed{b = 2}}$$

$\therefore -a - c = 0 \rightarrow a = c$ — (1)

$$l = \lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x\sin x} = 2$$

$\therefore a - b + c = 0$ — 2

eq^n 1 and 2 according,

$\therefore c - b + c = 0$

$\therefore 2c = b$ & $2a = b$

→ $$l = \lim_{x \to 0} \frac{ae^{x} + b\sin x - ce^{-x}}{x\cos x + \sin x} = 0$$

Appling L' Hospital rule,

$$\therefore \lim_{x \to 0} \frac{ae^{x} + b\cos x + ce^{-x}}{\cos x + x\sin x + \cos x} = 0$$

$$\therefore \frac{a + b + c}{2} = 2$$

$\therefore 2a + 2a = 4$

$\therefore \boxed{a = 1}$

$\therefore 2a = b$ $\therefore \boxed{b = 2}$

$\therefore 2c = b$ $\therefore \boxed{c = 1}$

7 $\lim_{x \to \frac{\pi}{2}} \dfrac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$

By L' Hospital,

$l = \lim_{x \to \frac{\pi}{2}} \dfrac{1/x - \pi/2}{\sec^2 x}$

$= \lim_{x \to \pi/2} \dfrac{\cos^2 \cancel{x} \pi/2}{x - \pi/2}$

$= \cancel{\lim_{x \to \pi/2}} = \lim_{x \to \pi/2} -2\cos\frac{\pi}{2} \cdot \sin\frac{\pi}{2}$

$= 0$

8 $\lim_{x \to 0} \left(\dfrac{a^x + b^x + c^x}{3}\right)^{1/\cancel{3}x3}$

$l = \lim_{x \to 0} \left[\dfrac{a^x + b^x + c^x}{3}\right]^{1/\cancel{3}x3}$

$\therefore \log l = \cancel{\lim_{x \to 0} \log \left[\dfrac{a^x + b^x + c^x}{3x \times 3}\right]} \quad \left(\dfrac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{\dfrac{3}{a^x + b^x + c^x} \cdot a^x \log a + b^x \log b + c^x \log c \,/3}{3}$

$$\therefore \log l = \frac{\log a + \log b + \log c}{9}$$

$$\therefore l = \frac{abc}{9}$$

9 $\lim_{x \to 0} [\cos x]^{\cot x}$

~~$l = \cot x \cdot \log(\cos x)$~~

$$l = \lim_{x \to 0} \cot x \cdot \log(\cos x)$$

$$= \lim_{x \to 0} \frac{\cot x}{1/\log(\cos x)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{-\operatorname{cosec}^2 x}{-\frac{\cos x}{\sin x}}$$

$$= \lim_{x \to 0} \tan x$$

$$= 1$$

5 $\lim_{x \to y} \left[\dfrac{x^y - y^x}{x^x - y^y} \right]$

$$l = \lim_{x \to y} \frac{y x^{y-1} - y^x \log y}{x^x(1+\log x) - 0}$$

$$= \lim_{x \to y} \frac{y \cdot y^{y-1} - y^y \log y}{y^y(1+\log y)}$$

$$= \frac{y^y(1-\log y)}{y^y(1+\log y)}$$

6 $\lim_{x \to 0} \dfrac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} \quad \left(\frac{0}{0}\right) \text{ form}$$

Appling L' Hospital rule,

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{2\sin x \cdot \cos x - 2x^2}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x^2} \quad \left(\frac{0}{0}\right) \text{ form}$$

Appling L' Hospital rule,

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$$= \lim_{x \to 0} \frac{e^{x} + e^{-x} - 2}{2\cos 2x - 2} \quad \left(\frac{0}{0}\right) \text{ form}$$

Appling L' Hospital,

$$= \lim_{x \to 0} \frac{e^{x} - e^{-x}}{-4\sin 2x} \quad \left(\frac{0}{0}\right) \text{ form}$$

Appling L' Hospital,

$$= \lim_{x \to 0} \frac{e^{x} + e^{-x}}{-8\cos 2x}$$

$$= \frac{2}{-8}$$

$$= \frac{-1}{4}$$

4

* Task 4 : Taylor's and Maclaurin's expansion.

1 Expand $\log(\cos x)$ about $\pi/3$ using Taylor's expansions.

given $f(x) = \log(\cos x)$

By Taylor's Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

put $a = \frac{\pi}{3}$

$$f\left(x - \frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right) + \frac{\left(x-\frac{\pi}{3}\right)^2}{2!}f''\left(\frac{\pi}{3}\right) + \frac{\left(x-\pi/3\right)^3}{3!}f'''\left(\frac{\pi}{3}\right) + \ldots$$

→ $f\left(\frac{\pi}{3}\right) = \log(\cos \pi/3) = \log 2$

→ $f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

$f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$

→ $F'''(x) = -\sec^2 x$

$F'''\left(\frac{\pi}{3}\right) = -4$

→ $F''''(x) = -2\sec x \cdot \tan x$

$F''''\left(\frac{\pi}{3}\right) = -2\sec\frac{\pi}{3} \cdot \tan\frac{\pi}{3}$

$= -2(2)\sqrt{3}$

$= -4\sqrt{3}$

→ $F(x) = \log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{4}{2}\left(x - \frac{\pi}{3}\right)^2$

$- \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3 + \cdots$

2 Expand $3x^3 - 2x^2 + x - 4$ in power of $(x+2)$

By Taylor's Series,

$F(x) = F(a) + (x-a)F'(a) + \frac{(x-a)^2}{2!}F''(a)$

$+ \frac{(x-a)^3}{3!}F'''(a) + \cdots$

Putting $a = -2$

$$F(x+2) = F(-2) + (x+2)F'(-2) + \frac{(x+2)^2}{2!}F''(-2)$$

$$+ \frac{(x+2)^3}{3!}F'''(-2) + \ldots$$

$\rightarrow$ $F(-2) = 3(-2)^3 - 2(-2)^2 - 2 - 4$

$\cancel{= -36 - 8}$

$= -24 - 8 - 6$

$= -38$

$\rightarrow$ $F'(x) = 9x^2 - 4x + 1$

$F'(-2) = 9(-2)^2 - 4(-2) + 1$

$= 36 + 8 + 1$

$= 45$

$\rightarrow$ $F''(x) = 18x - 4$

$F''(-2) = 18(-2) - 4$

$= 32 - 40$

$\rightarrow$ $F'''(x) = 18$

$F'''(-2) = 18$

$\rightarrow$ $F(x+2) = -38 + 45(x+2) - \frac{40}{2}(x+2)^2$

$$+ \frac{18}{6}(x+2)^3$$

$= -38 + 45(x+2) - 20(x+2)^2 + 3(x+2)^3$

3 Obtain Maclaurin series of $\sec x$.

By Maclurin's Series,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \ldots$$

$\rightarrow$ $F(x) = \sec x$

$F(0) = \sec 0 = 1$

$\rightarrow$ $F'(x) = \sec x \cdot \tan x$

$F'(0) = \sec 0 \cdot \tan 0 = 0$

$\rightarrow$ $F''(x) = \sec x \cdot \tan x \cdot \tan x + \sec x \cdot \sec^2 x$

$F''(0) = 1$

$\rightarrow$ $F'''(x) = \sec x \cdot \tan x \cdot \tan^2 x + 2\tan x \cdot \sec^2 x + 3\sec^2 x \cdot \sec x \cdot \tan x$

$F'''(0) = 0$

$\rightarrow$ $\cancel{F^{iv}(x)}$ $F'''(x) = \sec x \cdot \tan^3 x + 2\tan x \cdot \sec^2 x + 3\sec^3 x \cdot \tan x$

$\rightarrow$ $F^{iv}(x) = \sec x \cdot \tan x \cdot \tan^3 x + 3\tan^2 x \cdot \sec^2 x + 2\sec^2 x \cdot \sec^2 x + 2\tan x \cdot 2\sec x \cdot \sec x \cdot \tan x + 9\sec^2 x \tan x \cdot \sec x \cdot \tan x + 3\sec^3 x \cdot \sec^2 x$

$F^{iv}(x) = 5$

$$Sec(x) = 1 + x(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{5x^4}{4!} + \dots$$

$$Sec\,x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots$$

4 $$\log(Sec\,x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$$

let $\log(Sec\,x) = F(x)$

By Maclurin's Series,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2}F''(x) + \dots$$

→ $F(0) = 0$

→ $F'(x) = \frac{1}{Sec\,x} \cdot Sec\,x \cdot \tan x = \tan x$

$F'(0) = 0$

→ $F''(x) = Sec^2 x = 1 + \tan^2 x$

$F''(0) = 1$

→ $F'''(x) = 2Sec^2 x \cdot \tan x$

$F'''(0) = 0$

→ $F^{iv}(x) = 4Sec\,x \cdot \tan x \cdot \tan x + 2Sec^2 x \cdot Sec^2 x$

$F^{iv}(0) = 2$

→ $$\log(Sec\,x) = \cancel{x} + \frac{x^2}{2} + \frac{x^4}{12} + \dots$$