Unit - 4 Algebraic Structures And Morphism

* Task: 1 Binary Composition and its properties.

1 Define binary Operation with Example.

Let A is a non-empty set, then $a \times a$

$A \times A = \{(a,b) / a, b \in A\}$ then function

$* A \times A \rightarrow A$ is said binary operation A.
If $*(a, b) = a * b \in A$.

Ex.

$+$ is binary Operation on N, Z, R.
$-$ is binary Operation on N, Z, R
$-$ is not binary Operation on N.

2 Write all the properties of binary operations with Examples.

Let a, b, c a are non empty set one then, they follow this properties.

(1) Closure: a, b is close under the binary Operation.

Ex. $a, b \in G \Rightarrow a * b \in G$

(2) Associative: For all $a, b, c \in G$

$$a*(b*c) = (a*b)*c$$

(3) Exitance of Identity: They existies of element, $e \in G$, Such that,

For $\forall\ a, b, c \in G$

$$a*e = a = e*a$$

(4) Exitance of Inverse: For every $b \in G$, they every $e \in G$

$\forall\ a, b \in G$

$$a*b = e = \text{Identity}.$$

Ex. $(Z, *)$, $\forall a, b \in Z$, $a * b = a + b + 5$.

Let a, b is non-empty set on Z then,

(1) Closure:

For $\forall a, b \in Z$, $a * b = a + b + 5 \in Z$

So, $\langle Z, * \rangle$ is closure under the binary Operation.

(2) Associative:

For $\forall a, b, c \in Z$,

$$(a * b) * c = (a+b+5) * c$$
$$= a + b + c + 10 \quad — (1)$$

$$(a) * (b * c) = a * (b + c + 5)$$
$$= a + b + c + 10 \quad — (2)$$

From eq^n, 1 = 2 $\langle Z, * \rangle$ is Associative underthe operation.

(3) Exitance of Identity:

For $\forall a, b, c \in Z$ and $e \in Z$

$\therefore a * e = a$

$\therefore a + e + 5 = a$

$\therefore e = -5 \in Z$

So, $\langle Z, * \rangle$ has identity element -5.

(4) Exitance of Inverse:

$\forall, a, b \in Z$

$b = a^{-1}$

$a * b = e$

$\therefore a + b + 5 = -5$

$\therefore a + b = -10 \in Z$

$\therefore a^{-1} = -10 - a \in Z$

So, $\langle Z, * \rangle$ is follow the all the properties.

3 Test the Following Properties For the given structure.

Properties: Closure, Identity, Inverse.

(a) $(Z, *)$, $\forall a, b \in Z$, $a * b = a + b + 5$

Let Z is not set empty set and $a, b \in Z$

(1) Closure:

For $\forall$ $a, b \in Z$, $a * b = a + b + 5 \in Z$

So, $\langle Z, * \rangle$ is follow closure properties.

(2) Identity:

For $\forall$, $a, e \in Z$,

$$a * e = a$$
$$\therefore a + e + 5 = a$$
$$\therefore e = -5 \in Z$$

For $\langle Z, * \rangle$, identity element is -5.

(3) Inverse:

For $\forall$, $a, b \in Z$

$$\therefore a * b = e$$
$$\therefore a + b + 5 = -5$$
$$\therefore a = -10 + b \in Z$$

For $\langle Z, * \rangle$, Inverse propertie is satisfy.

(b) $(Z^+, *)$, $\forall a, b \in Z^+$, $a * b = a^b$

Let, Z^+ is not empty set and $a, b \in Z^+$.

(1) Closure:

For $\forall a, b \in Z^+$, $a * b = a^b \in Z^+$

(2) Existance of Identity:

For $\forall a, b \in Z^+$ and $e \in Z^+$

$\therefore \quad a * e = a$

$\therefore \quad a^e = a$

$\therefore \quad e = 1 \in Z^+$

(3) Existance of Inverse:

For $\forall a, b \in Z^+$,

$a * \cancel{a} = e$

$\therefore \quad a^{\cancel{a}^{-1}} = 1$

$\therefore \quad a \cdot \log a^{-1} = 1$

$\log a^{-1} = \dfrac{1}{a}$

(4) Associative:

For, $a, b, c \in Z^+$

$(a * b) * c = a^b * c$
$= (a^b)^c \in Z^+$ — ①

$a * (b * c) = a * b^c$
$= a^{b^c} \in Z^+$ — ②

(c) $(R, *)$, $\forall\, a, b \in R$, $a * b = \frac{a}{a+b}$

Let R is non empty set and $a, b \in R$.

(1) Closure:

For $\forall$, $a, b \in R$,
$a * b = \frac{a}{a+b} \in R$.

(2) Associative:

For, $a, b, c \in Z^+$

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$$(a * b) * c = \frac{a}{a+b} * c$$

$$= \frac{a}{a+b+c} \quad \text{— (1)}$$

$$a * (b * c) = a * \frac{b}{b+c}$$

$$= \frac{a}{b/b+c}$$

$$= \frac{a(b+c)}{b} \quad \text{— (2)}$$

From eq^n $1 \neq 2$
$\langle R, * \rangle$ not follow Associative

(3) Existance of Identity:

For, $a, b \in R$, $e \in R$

$\therefore a * e = a$

$\therefore \frac{a}{a+e} = a$

$\therefore a + e = 1$

$\therefore e = 1 - a \in R$.

For $\langle R, * \rangle$ identity element is $1-a$.

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(4) Existance of Inverse:

Let, $a, a^{-1} \in R$

$\therefore a * a^{-1} = e$

$\therefore \dfrac{a}{a+a^{-1}} = 1-a$

$\therefore \dfrac{a}{1-a} - a = a^{-1}$

$\therefore a^{-1} = \dfrac{a}{1-a} - a \in R$

For $\langle R, * \rangle$, Inverse of a is $\dfrac{a}{1-a} - a$

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* Task 2 : Various Algebriac Structures and Order of element.

1 Define the following terms.

(a) Semi Group:

Let, G is non empty set and (G, *) is compozition * is Follow Associative then, G is said to be Semi Group.

$$(a * b) * c = a * (b * c)$$

(b) Monoid:

Let, G is non empty set and (G, *) is existes an identity element then, G is said to be Monoid.

$$\therefore a * e = a$$

(c) Group:

Let, G is non empty set and (G, *) is follow this properties, then G is said to be Group.

For a, b, c G G

ures

1) Closure : $a + b \in G$
2) Associative : $a * (b * c) = (a * b) * c$
3) Identity : $a * e = a$
4) Inverse : $a * a^{-1} = e$.

(d) Abelian Group:

A Group $(G, *)$ is said to be Abelian Group,

If, $\forall\ a, b \in G$, $a * b = b * a$

G is follow the Commulative Group Property.

(e) Order of the Group:

Let $(G, *)$ be a Group, then numbter of element in G is said to be order of Group.

(f) Order of element : The order of an element g in A Group G is the smallest positive integer such that,

$$g^n = e$$

The Order of an element G is donated by $O(g)$.

2 Prove that set of positive rational number forms an Abelian Group under the compostion of
$$a * b = \frac{ab}{2}$$

=> For, abelian Group, Commulative Group Property must be follow.

$\therefore a * b = \frac{ab}{2} \in R$ — ①

$\therefore b * a = \frac{ba}{2} = \frac{ab}{2} \in R$ — ②

From, eq^n 1 = 2

So, Set of Rational ~~P~~ number is said to Abelian Group.

3 Check whether (R, X) is Group or not.

Let, R is non empty set and $a, b, c \in R$.

For be a Group, R is must follow this four Properties.

1 Closure:

For $a, b \in R$,

$\therefore a+b \in R$

So, R satisfy closure Properties.

2 Associative:

For $a, b, c \in R$,

$(a * b) * c = a \cdot b * c$
$= abc$ — (1)

$a * (b * c) = a * bc$
$= abc$ — (2)

From eq^n 1 = 2.
So, R is satisfy Associative Property.

3 Exitance of ~~Group~~ Identity:

For $a, b, c \in R$

$a * e = a$
$\therefore ae = a$
$\therefore e = 1$

For R, Identity element is 1.

4 Existance of Inverse:

For $a, e \in R$

$\therefore a * a^{-1} = e$
$\therefore aa^{-1} = 1$

$\therefore a^{-1} = \frac{1}{a}$

Here, R is satisfy the all the Properties.
Hence, R is a Group.

4 Classify the algebraic structure for,

(a) $(Z_7 - \{0\}, \times_7)$

$Z_7 = \{[1], [2], [3], [4], [5], [6]\}$

(i) Closer: From Composition table, For every $[i][j] \in Z_7$.

(ii) Associative: From Composition table, it is clear that Z_7 is associative.

→ Composition Group:

$\times_7$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

(iii) Identity: From Composition table, identity elements is [1].

(iv) Inverse: From Composition table,

Inverse of [1] is [1].
Inverse of [2] is [4].
Inverse of [3] is [5].
Inverse of [4] is [2].
Inverse of [5] is [3].
Inverse of [6] is [6].

Hence, $(Z_7 - \{0\}, \times_7)$ is algebraic structure.

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(2) (M_{22}, X); where M_{22} is set of all 2 x2 Matrices.

Let M_{22} matrix in non seempty set of matrix.

and,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = \begin{bmatrix} i & j \\ K & l \end{bmatrix} \in M_{22}$$

(1) Closure : Let $A, B \in M_{22}$

$$\therefore A+B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in M_{22}$$

(2) Associativity:

$$\rightarrow A * (b * C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ci+fk & ej+Kl \\ gi+hk & gj+hl \end{bmatrix}$$

$$= \begin{bmatrix} aei+afk+bgi+bhk & aei+afl+bgi+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgi+dhl \end{bmatrix} \quad -①$$

$\rightarrow$ $(A * B) * C = \begin{bmatrix} ae+by & af+bh \\ ce+dy & cf+hd \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$

$= \begin{bmatrix} aei+bgi+afk+bhk & aef+bgi+afe+bhe \\ cei+dgi+cfk+dhk & cej+dgi+cfe+hde \end{bmatrix}$ —(2)

From, eq^n $1 = 2$

(3) Identity:

Let $e = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \in M_{22}$

$\therefore A * e = A$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} = \begin{bmatrix} a & \cancel{d}\, b \\ c & \cancel{d}\,\cancel{b}\, d \end{bmatrix}$

$\therefore e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Identity element is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(4) Inverse:

Let $a * a^{-1} = e$

$$a^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$a^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence, M_{22} is a Algebric structre.

5 Prove that the set of cubic root of unity under multiplication is Abelian group. Find the order of the group and order of each element.

Here, we want to show that cubic roots of unity under multiplication. For that,

$$\therefore x^3 = 1$$

$$\therefore x^3 - 1 = 0$$

$$\therefore (x-1)(x^2+x+1) = 0$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore x = 1, w, w^2$$

Hence, $G = \{1, w, w^2\}$

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→ Composition table:

$*$	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

(1) Closer: From Composition table, For every $a * b \in G$, $\forall a, b \in G$

(2) Associativity: From Composition table, it is clear that G is associative.

(3) Identity: From Composition table, identity element is 1.

(4) Inverse: From Composition table,

Inverse of 1 is 1.
Inverse of w is w^2
Inverse of w^2 is w.

Hence, Given set G is group under the multiplication.

6 Find the order of the group and Order of each element For,

(a) $(Z_7 - \{0\}, \times_7)$

$Z_7 = \{[1], [2], [3], [4], [5], [6]\}$

Order of Group = 6

$g = [1] = [1]^1 \Rightarrow O([1]) = 1$

$g = [2] = [2]^3 \Rightarrow O([2]) = 3$

$g = [3] = [3]^6 \Rightarrow O([3]) = 6$

$g = [4] = [4]^4 \Rightarrow O([4]) = 4$

$g = [5] = [5]^3 \Rightarrow O([5]) = 3$

$g = [6] = [6]^2 \Rightarrow O([6]) = 2$

(b) $(Z_5, +_6)$

$Z_5 = \{[1], [2], [3], [4], [0]\}$

Order of Group = 5

$g = [1] = [1]^5 \Rightarrow O([1]) = 5$

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$g = [2] = [2]^5 \Rightarrow O([2]) = 5$

$g = [3] = [3]^5 \Rightarrow O([3]) = 5$

$g = [4] = [4]^5 \Rightarrow O([4]) = 5$

$g = [0] = [0]^5 \Rightarrow O([0]) = 5$

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* Task : 3 : Sub Group and Coset of a Sub Group.

1 Define following Terms.

(i) Subgroup

Let G is a Group under the binary operation and a ⊆ G If a is called subgroup of G, then a subgroup follow all the propertiy of Group.

(ii) Improper Sub Group:

Let G is a Group under the binary operation and G has to SubGroup which is Group itself and identity, they Group are called Improper Sub Group.

(iii) Coset:

Let H be a Subgroup of G then the set $\{a * h\}; h \in H\}$ is called the left coset generated by a and H. It is denoted by aH.

Left Coset = a+H

Right Coset = H+a

(iv) Index of Sub group:

Let G is a Group and H is a subgroup, then number of the coset of subgroup is known as Index of Subgroup.

2 State Lagrange's Theorem with the condition.

-> Lagrange's Theorem:

The order of each sub group a finite Group G is a divider of Order of G.

Or

Let G be a Group with O(G) = n and H be a subgroup with O(H) = n then O(H) / O(G).

- Condition:

This are the two condition of Lagrange's Theorem.

1 $a, b \in H \Rightarrow a * b \in H$

2 $b \in H \Rightarrow b^{-1} \in H$

3 Find all Subgroups of the following groups:

(a) $(Z_6, +_6)$

$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$

- Composition table:

$+_6$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

From the table,

1 Closure: $\forall a, b \in Z_6$
$a +_6 b \in Z_6$

2 From the table, We can see that $(Z_6, +_6)$ is associative.

3 Identity : 0 is the identity element of $(Z_6, +_6)$.

4 Existence of Inverse :

Inverse of 0 is 0.
Inverse of 1 is 5 and inverse of 5 is
Inverse of 2 is 4
Inverse of 3 is 3
Inverse of 5 is 1

$(Z_6, +_6)$ is follow all the Property of Group.

- Subgroup :

$(H_3, +_6)$ is a Subgroup of $\{0, 1, 3\}$

$(H_4, +_6)$ is a Subgroup of $\{0, 2, 4\}$

$(H_5, +_6)$ is a Subgroup of $\{0, 5\}$

(b) $G = (\{\pm 1, \pm i\}, \times)$

$G = \{1, -1, +i, -i\}$

- Composition table:

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

From the table,

(i) Closure: $\forall a, b \in G, \ a \times b \in G$

(ii) From the table, we can see that G is a Associative.

(iii) From the table, Identity element is 1.

(iv) Inverse:

Inverse of 1 is 1
Inverse of -1 is -1
Inverse of -i is +i
Inverse of i is -i

- Sub Group:

$(H_3, \times)$ where $H = \{1, -1\}$

~~$(H_4, \times)$ where $H = \{1, i, -i\}$~~

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4 Prove that H is sub group of a give Group G in each of the following. Find all coset of H. also Find index of H (G:H).

(i) $G = (Z_6, +_6)$ and $H = (0, 3)$

=) For Group $G = (Z_6, +_6)$

Composition table:

$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$

$+_6$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

From the table,

(i) Closure: $\forall a, b \in G$, $a +_6 b \in G$

(ii) From the table, we can see that $(Z_6, +_6)$ is associative.

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(iii) Identity element is [0].

(iv) Inverse:

Inverse of 0 is 0
Inverse of 1 is 5
Inverse of 2 is 4
Inverse of 3 is 3
Inverse of 4 is 2
Inverse of 5 is 1

Here, $(Z_6, +_6)$ is a Group.

=) For $H = \{0, 3\}$

$+_6$	0	3
0	0	3
3	3	0

From the table,

(i) Closure: $\forall a, b \in H,\ a +_6 b \in H$

(ii) From the table, we can see that, $(H, +_6)$ is associative.

(iii) Identity Element is [0].

(iv) Inverse of 0 is 0
Inverse of 3 is 3

So, $(H, +_6)$ is a SubGroup of $(Z_6, +_6)$.

$\Rightarrow$ For Coset

- For $0 \in Z_6$

Left coset $= \{0 +_6 0,\ 0 +_6 3\}$
$= \{0, 3\}$

Right coset $= \{0 +_6 0,\ 3 +_6 0\}$
$= \{0, 3\}$

- For $1 \in Z_6$

Left coset $= \{1 +_6 0,\ 1 +_6 3\}$
$= \{1, 4\}$

Right coset $= \{0 +_6 1,\ 3 +_6 1\}$
$= \{1, 4\}$

- For $2 \in Z_6$

Left coset $= \{2 +_6 0,\ 2 +_6 3\}$
$= \{2, 5\}$

Right coset $= \{0 +_6 2,\ 3 +_6 2\}$
$= \{2, 5\}$

- For $3 \in Z_6$

Left coset $= \{3 +_6 0, 3 +_6 3\}$
$= \{3, 0\}$

Right Coset $= \{0 +_6 3, 3 +_6 3\}$
$= \{3, 0\}$

- For $4 \in Z_6$

Left coset $= \{4 +_6 0, 4 +_6 3\}$
$= \{4, 1\}$

Right coset $= \{0 +_6 4, 3 +_6 4\}$
$= \{4, 1\}$

- For $5 \in Z_6$

Left coset $= \{5 +_6 0, 5 +_6 3\}$
$= \{5, 2\}$

Right coset $= \{0 +_6 5, 3 +_6 5\}$
$= \{5, 2\}$

Here, Distinct cosets $= 3$

Index of H $[G : H] = 3$

(ii) $G = (\{\pm 1, \pm i\}, \times)$, $H = (\{1, -1\}, \times\}$

=> For $G = \{+1, -1, i, -i\}$

Composition table:

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

From the table,

(i) Closure: $\forall a, b \in G$, $a \times b \in G$

(ii) From the table we can see that G is a Associative.

(iii) Identity element is 1.

(iv) Inverse:

Inverse of 1 is 1
Inverse of -1 is -1
Inverse of i is -i
Inverse of -i is i

Here, All the Property is follow.
So, G is a Group.

=> For $H = (\{1, -1\}, \times)$

$\times$	1	-1
1	1	-1
-1	-1	1

From the table,

(i) Closure : $\forall\ a, b \in H,\ a \times b \in H$

(ii) From the table, we can see that H is associative.

(iii) Identity element is 1.

(iv) Inverse :

Inverse of 1 is 1
Inverse of -1 is -1

Hence, H is a Subgroup of G.

=> For Coset:

- For $1 \in G$

Left Coset $= \{1 \times 1,\ 1 \times -1\}$
$= \{1, -1\}$

Right Coset = $\{1 \times 1, -1 \times 1\}$
$= \{1, -1\}$

- For $-1 \in G$

Left Coset = $\{-1 \times 1, -1 \times -1\}$
$= \{-1, 1\}$

Right Coset = $\{1 \times -1, -1 \times -1\}$
$= \{-1, 1\}$

- For $i \in G$

Left Coset = $\{i \times 1, i \times -1\}$
$= \{i, -i\}$

Right Coset = $\{1 \times i, -1 \times i\}$
$= \{i, -i\}$

- For $-i \in G$

Left Coset = $\{-i \times 1, -i \times -1\}$
$= \{-i, i\}$

Right Coset = $\{1 \times -i, -1 \times -i\}$
$= \{-i, i\}$

Here, Distinct Coset = 2

Index of H $(G:H) = 2$

(iii) $G = (Z, +)$ and $H = (2Z, +)$

=> For Group G

$G = \{ ---, -1, 0, 1, 2, --- \}$

(i) Closure : $\forall a, b \in G, \quad a+b \in G;$

(ii) Associative : $\forall a, b, c \in G$

$\therefore (a+b)+c = a+(b+c) \in G$

$\therefore$ It is Associative.

(iii) Identity element is 0.

(iv) Inverse : $\forall a \in G$

$a+b = e$
$\therefore a+b = 0$
$\therefore b = -a$

So, All Property is follow by the G So, G is a Group.

=> For Subgroup H

$H = \{ ---, -2, 0, 2, 4, 6, --- \}$

(i) Closure : $\forall a, b \in H,\ a+b \in H$

(ii) Associative : $\forall a, b, c \in H$

$\therefore (a+b)+c = a+(b+c)$

$\therefore$ It is associative

(iii) Identity element is 0

(iv) Inverse : $\forall a \in H$

$a+b = e$

$\therefore a+b = 0$

$\therefore a = -b$

So, Here, H is a Subgroup of G.

$\Rightarrow$ Coset:

- For $0 \in Z$

Left Coset - H

$0 + H = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

Right Coset

$H + 0 = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

- For $1 \in Z$

L.C $= 1 + H = \{\ldots, -3, -1, 1, 3, \ldots\}$

R.C $= H + 1 = \{\ldots, -3, -1, 1, 3, \ldots\}$

- For $2 \in Z$

L.C $= 2 + H = \{\ldots, -2, 0, 2, \ldots\}$

R.C $= H + 2 = \{\ldots, -2, 0, 2, \ldots\}$

- For $3 \in Z$

L.C. $= 3 + H = \{\ldots, -3, -1, 1, 3, \ldots\} = H + 1$

R.C. $= H + 3 = \{\ldots, -3, -1, 1, 3, \ldots\} = H + 1$

- For $4 \in Z$

L.C. $= 4 + H = \{\ldots, -2, 0, 2, \ldots\} = H + 2$

R.C. $= H + 4 = \{\ldots, -2, 0, 2, \ldots\} = H + 2$

- Distinct Coset $= \{H, 1 + H, 2 + H\}$

- Index of $H (G : H) = 3$

* Task - 4 : Various Group - 1

1 Define the following terms.

(i) Permutation Group

A Permutation Group is a finite a Group G whose elements are permutation of a given set and whose group operation in composition of Permutation of G.

(ii) Cyclic Group:

Let G be a Group and $a \in G$ then, For some $a \in G$ every element of G is of the from a^n where n is some integer that is,

$$G = \{a^n, n \in z\}$$

(iii) Generator:

The element those Generates a cyclic Group, this elements are called Generator.

2 State Cayley's Theorem. Prove that (S_3, o) is non-abelian permutation Group.

Cayley's Theorem: If G is a Group then there exists a subgroup H of $Sym(G)$ such that G is isomorphic to H.

→ $S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

Composition table.

o	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_5	P_6	P_5	P_4	P_3
P_3	P_3	P_5	P_1	P_6	P_2	P_4
P_4	P_4	P_6	P_5	P_1	P_3	P_2
P_5	P_5	P_3	P_4	P_2	P_6	P_1
P_6	P_6	P_4	P_2	P_3	P_1	P_5

From the table,

(i) Closure: $\forall P_i, P_j \in S_3 \rightarrow P_i \cdot P_j \in S_3$

(ii) Associative: From the table, it is clear that S_3 is associative.

(iii) Identity: P_1 is the identity element.

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(iv) Inverse : From table inverse of $P_1, P_2, P_3, P_4, P_5, P_6$ are $P_1, P_2, P_3, P_4, P_5, P_6$ respectvely.

(v) Let $P_3, P_4 \in S_3$

$$So, \quad P_3 \cdot P_4 = P_5 \text{ and } P_4 \cdot P_3 = P_6$$

$$\therefore P_3 \cdot P_4 \neq P_4 \cdot P_3$$

Hence, S_3 is not a abelian Group.

3 Prove that following structure is cyclic. Find all its generators.

(i) $G = (\{1, -1, i, -i\}, \times)$

Here, $a = -1$ and $a = 1$ is not contain the all Group element such that $a = -1$ and $a = 1$ is not generators.

$a = i \Rightarrow (i)^1 = i$
$(i)^2 = -1$
$(i)^3 = -1$
$(i)^4 = 1$

$a = -i \Rightarrow (-i)^1 = -i$
$(-i)^2 = -1$
$(-i)^3 = i$
$(-i)^4 = 1$

$a = -i$ and $a = i$ contain all Group element by tacking different power.

Such that $a = i$ and $-i$ is Generator.

$\therefore \langle G, X \rangle$ is a Cyclic Group generated by i and $-i$

(ii) $(G = Z_8, +_8)$

$G = \{[0], [1], [2], [3], [4], [5], [6], [7]\}$

- For $[0] \in G$

$\therefore 0^1 = 0$
$\therefore 0^2 = 0$

$[0]$ is not Generator.

- For $[1] \in G$

$\therefore [1]^1 = [1]$
$\therefore [1]^2 = [2]$
$\therefore [1]^3 = [3]$
$\therefore [1]^4 = [4]$
$\therefore [1]^5 = [5]$
$\therefore [1]^6 = [6]$
$\therefore [1]^7 = [7]$
$\therefore [1]^0 = [0]$

$\therefore$ [1] is Generator.

- For $[2] \in G$

$\therefore [2]^1 = [2]$
$\therefore [2]^2 = [4]$
$\therefore [2]^3 = [6]$
$\therefore [2]^0 = [0]$

$\therefore$ [2] is not Generator.

- For $[3] \in G$,

$\therefore [3]^1 = [3]$
$\therefore [3]^2 = [6]$
$\therefore [3]^3 = [1]$
$\therefore [3]^4 = [4]$
$\therefore [3]^5 = [7]$
$\therefore [3]^6 = [2]$
$\therefore [3]^7 = [5]$
$\therefore [3]^8 = 0$

$\therefore$ [3] is Generator.

- For $[4] \in G$,

$\therefore [4]^1 = 4$
$\therefore [4]^2 = 0$
$\therefore [4]^3 = 4$

$\therefore$ [4] is not Generator.

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- For $[5] \in G$

$\therefore [5]^1 = [5]$
$\therefore [5]^2 = [2]$
$\therefore [5]^3 = [7]$
$\therefore [5]^4 = [4]$
$\therefore [5]^5 = [1]$
$\therefore [5]^6 = [6]$
$\therefore [5]^7 = [3]$
$\therefore [5]^8 = [0]$

$\therefore$ [5] is a Generator.

- For $[6] \in G$

$\therefore [6]^1 = [6]$
$\therefore [6]^2 = [4]$
$\therefore [6]^3 = [2]$
$\therefore [6]^4 = [0]$

$\therefore$ [6] is not Generator

- For $[7] \in G$

$\therefore [7]^1 = [7]$
$\therefore [7]^2 = [6]$
$\therefore [7]^3 = [5]$
$\therefore [7]^4 = [4]$
$\therefore [7]^5 = [3]$
$\therefore [7]^6 = [2]$
$\therefore [7]^7 = [1]$

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$\therefore [7]^0 = 0$

$\therefore [7]$ is not Generator.

$\therefore \langle Z_7, +_8 \rangle$ is a Cyclic Group generator by 1, 5, 7, 3.

4 Find all generators of a cyclic group of Order 16.

By Lagrange's Theorem

$H = \{1, 2, 4, 8, 16\}$

$\rightarrow$ If $O(H) = 1 \Rightarrow H = \langle a^{16/1} \rangle$

$H_1 = \{a^{16}\}$

$\rightarrow$ If $O(H) = 2 \Rightarrow H = \langle a^{16/2} \rangle$

$H_2 = \{a^{16}, a^8\}$

$\rightarrow$ If $O(H) = 4 \Rightarrow H = \langle a^{16/4} \rangle$

$H_3 = \{a^{16}, a^8, a^{12}, a^4\}$

$\rightarrow$ If $O(H) = 8 \Rightarrow H = \langle a^{16/8} \rangle$

$H_3 = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}\}$

$\rightarrow$ If $O(H) = 16 \Rightarrow H = \langle a^{16/16} \rangle$

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$H_4 = \{a^1, a^2, a^3, a^4, a^5, a^6, a^7,$
$a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14},$
$a^{15}, a^{16}\}$

-) Now we Find Order of each element,

$a^1 = (a^1)^{16} = e \Rightarrow O(a) = 16$
$a^2 = (a^2)^8 = e \Rightarrow O(a^2) = 8$

$a^3 = (a^3)^{16} = e \Rightarrow O(a^3) = 16$
$a^4 = (a^4)^4 = e \Rightarrow O(a^4) = 4$

$a^5 = (a^5)^{16} = e \Rightarrow O(a^5) = 16$
$a^6 = (a^6)^8 = e \Rightarrow O(a^6) = 8$

$a^7 = (a^7)^{16} = e \Rightarrow O(a^7) = 16$
$a^8 = (a^8)^2 = e \Rightarrow O(a^8) = 2$

$a^9 = (a^9)^{16} = e \Rightarrow O(a^9) = 16$
$a^{10} = (a^{10})^8 = e \Rightarrow O(a^{10}) = 8$

$a^{11} = (a^{11})^{16} = e \Rightarrow O(a^{11}) = 16$
$a^{12} = (a^{12})^4 = e \Rightarrow O(a^{12}) = 4$

$a^{14} = (a^{14})^8 = e \Rightarrow O(a^{14}) = 8$
$a^{13} = (a^{13})^{16} = e \Rightarrow O(a^{13}) = 16$

$a^{15} = (a^{15})^{16} = e \Rightarrow O(a^{15}) = 16$
$a^{16} = (a^{16})^1 = e \Rightarrow O(a^{16}) = 1$

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Here, $a^1, a^5, a^7, a^{11}, a^{13}$ are Generator of Grop G.

5 IF σ, μ and $\tau \in S_6$ defined by

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$, $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 4 \end{pmatrix}$

and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$.

Find the following.

(1) $\sigma^2 \tau$

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$\sigma^2 \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 5 & 1 \end{pmatrix}$$

(2) $\mu^{-1}\tau$

$\therefore \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ c & d & b & f & e & a \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

$\therefore \mu^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 5 & 4 \end{pmatrix}$

$\therefore \mu^{-1}\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 4 & 5 & 2 \end{pmatrix}$

(3) $O(\sigma)$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$

Cycle form $\rightarrow$ $(1\ 2\ 3)(4)(5\ 6)$

For order : $LCM(3, 1, 2)$

$O(\sigma) = 6$

(4) $O(\tau)$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

Cycle = $(1\ 6)(2\ 4)(5)(3)$
For Order = $LCM(2\ 2\ 1\ 1)$
$O(\tau) = 2$

=) Verify:

1 $(\sigma\mu)^{-1} = \mu^{-1}\sigma^{-1}$

L.H.S. $= \sigma\mu$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 6 & 1 & 5 \end{pmatrix}$$

$\therefore (\sigma\mu)^{-1} =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 6 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$\therefore (\sigma \mu)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{pmatrix}$ — (1)

RHS :

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \cancel{8}\,3 & \cancel{2}\,1 & \cancel{3}\,2 & 4 & 6 & 5 \end{pmatrix}$

$\cancel{\sigma^{-1} \mu^{-1}}$

$\mu^{-1} \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{pmatrix}$ — (2)

$\therefore$ LHS = RHS.

Hence, $(\sigma \mu)^{-1} = \mu^{-1} \sigma^{-1}$

2 $\sigma \cdot (\mu \cdot \tau) = (\sigma \cdot \mu) \cdot \tau$

LHS.

$\mu \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$

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$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$\sigma \circ (\mu \circ \tau) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \circ$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 6 & 1 \end{pmatrix}$$

R.H.S. :

$$\sigma \circ \mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(\mu(1)) & \sigma(\mu(2)) & (\sigma \mu(3)) & \sigma \mu(4) \end{pmatrix.$$

$$\begin{matrix} 5 & 6 \\ \sigma(\mu(5)) & \sigma(\mu(6)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{pmatrix} = a$$

$$(\sigma \circ \mu) \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a(\tau(1)) & a(\tau(2)) & a(\tau(3)) & a(\tau(4)) \end{pmatrix.$$

$$\begin{matrix} 5 & 6 \\ a(\tau(5)) & a(\tau(6)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 6 & 1 \end{pmatrix}$$

Here, LHS = RHS.

Hence, $\sigma \circ (\mu \circ \tau) = (\sigma \circ \mu) \circ \tau$ is prove.

* Task : 5 : Various Groups - II

1 Define the following term

(i) Normal Sub Group:

A Subgroup H of a Group G is said to be a normal sub Group of G if

Left Coset = Right Coset

$\therefore \cancel{aH}\ Ha = aH$

Every abilian Group is a normal Subgroup.

(ii) Quotient Group:

If H is a normal subgroup of G then the set of all left coset of G forms a group with respest to multiplication left coset and defined as,

$$(aH)b(H) = ab(H)$$

(2) Prove that Sub group $H = \{P_1, P_2, P_3\}$ is normal for the group (S_3, o)

For Finding the Subgroup.

o	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_6	P_5	P_4	P_3
P_3	P_3	P_5	P_1	P_6	P_2	P_4
P_4	P_4	P_6	P_5	P_1	P_3	P_2
P_5	P_5	P_3	P_4	P_2	P_6	P_1
P_6	P_6	P_4	P_2	P_3	P_1	P_5

→ For normal Group, Always,

Right Coset = Left Coset

- ~~Left Coset~~

- For $P_1 \in H$

$$\text{Right coset} = (P_1, P_2, P_3) \cdot P_1$$
$$= (P_1 \cdot P_1,\ P_2 \cdot P_1,\ P_3 \cdot P_1)$$
$$= (P_1, P_2, P_3) \quad \text{— (1)}$$

$$\text{Left Coset} = P_1 \cdot (P_1, P_2, P_3)$$
$$= (P_1 \cdot P_1,\ P_1 \cdot P_2,\ P_1 \cdot P_3)$$
$$= (P_1, P_2, P_3) \quad \text{— (2)}$$

eq^n 1 = 2

→ For $P_2 \in \cancel{\epsilon}\, G$

Right Coset $= (P_1, P_2, P_3) \cdot P_2$
$= (P_2, P_1, P_6)$ — ①

Left Coset $= P_2 \cdot (P_1, P_2, P_3)$
$= (P_2, P_1, P_6)$ — ②

$eq^n\ 1 = 2$

→ For $P_3 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_3$
$= (P_3, P_6, P_1)$ — ①

Left coset $= P_3 (P_1, P_2, P_3)$
$= (P_3, P_6, P_1)$ — ②

$eq^n\ 1 = 2$

- For $P_4 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_4$
$= (P_4, P_6, P_5)$ — ①

Left coset $= P_4 \cdot (P_1, P_2, P_3)$
$= (P_4, P_6, P_5)$ — ②

$eq^n\ 1 = 2$

- For $P_5 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_5$
$= (P_5, P_3, P_4)$ — ①

Left coset $= P_5 \cdot (P_1, P_2, P_3)$
$= (P_5, P_3, P_4)$ — ②

- For $P_6 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_6$
$= (P_6, P_4, P_2)$ — ①

Left coset $= P_6 \cdot (P_1, P_2, P_3)$
$= (P_6, P_4, P_2)$ — ②

Here, ~~G~~ H is be a normal sub group of G

$\cancel{G/H = \{H1, H2, H3, H4, H5, H6\}}$

$G/H = \{HP_1, HP_2, HP_3, HP_4, HP_5, HP_6\}$

- Check whether Sub Group $H = \{P_1, P_4\}$ is normal for the Group (S_3, o)?

For normal Group Always,

We have to check,

Left Coset = Right Coset.

- For $P_1 \in G$

~~Right Coset~~ $= \cancel{(P_1, P_2, P_3) \cdot P_1}$

$= \cancel{(P_1, P_2, P_3)}$ — (1)

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~~Left coset = $P_1 \cdot (P_1$~~

Right coset $= (P_1, P_4) \cdot P_1$
$= (P_1, P_4)$ — ①

Left Coset $= P_1 \cdot (P_1, P_4)$
$= (P_1, P_4)$ — ②

$eq^n \; 1 = 2$

- For $P_2 \in G$

Right coset $= (P_1, P_4) \cdot P_2$
$= (P_2, P_5)$ — ①

Left Coset $= P_2 \cdot (P_1, P_4)$
$= (P_3, P_6)$ — ②

$eq^n \; 1 \neq 2$

Here, For P_2, $H = \{P_1, P_4\}$ is not be a normal subgroup

Thus, $H = \{P_1, P_4\}$ is not be a normal subgroup of $(S_3, \circ)$

3 Prove that Sub Group $H = (1, -1)$ is normal for the Group $G = (\{1, -1, i, -i\}, \times)$

→ Composition table.

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-1	i	1	-1

→ For normal Group,

Left Coset = Right Coset.

- For $1 \in G$

Left Coset $= 1 \cdot (1, -1)$
$= (1, -1)$ — ①

Right Coset $= (1, -1) \cdot 1$
$= (1, -1)$ — ②

$eq^n \; 1 = 2$

- For $-1 \in G$

Left coset $= -1 \cdot (1, -1)$
$= (-1, 1)$ — ①

સંયમઅને સાદગી દ્વારા જીવનમાં શાંતિ અને સંતોષ અનુભવાય છે.

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Right Coset = $(1, -1) \cdot -1$
$= (-1, 1)$ — 2

eq^n 1 = 2

- For $i \in G$

Left Coset = $i \cdot (1, -1)$
$= (i, -i)$ — 1

Right coset = $(1, -1) \cdot i$
$= (i, -i)$ — 2

eq^n 1 = 2

- For $-i \in G$

Left Coset = $-i (1, -1)$
$= (-i, i)$ — 1

Right Coset = $(1, -1) \cdot -i$
$= (-i, i)$ — 2

eq^n 1 = 2

Here, For all the Group G
Left Coset = Right coset

Thus, $H = \{1, -1\}$ is a normal subgroup For ~~$\in S_3$~~ G.

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4 Prove that G/H is Quotient Group For the following Group G and normal subgroup H, Using composition table,

(i) $G = (\{\pm 1, \pm i\}, \times)$ $H = (\{1, -1\}, \times)$

-> Composition table For G

$G = \{-1, 1, -i, i\}$

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Composition table For H

$H = \{1, -1\}$

$\times$	1	-1
1	1	-1
-1	-1	1

-> For Normal Subgroup,

Left Coset = Right Coset

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→ For $1 \in G$

Left coset $= 1 \cdot \{1, -1\}$
$= (1, -1)$
$= (1, -1) \cdot 1$
$=$ Right coset

→ For $-1 \in G$

Left Coset $= -1 \cdot (1, -1)$
$= (-1, 1)$
$= (1, -1) \cdot -1$
$=$ Right Coset

→ For $i \in G$

Left coset $= i \cdot (1, -1)$
$= (i, -i)$
$= (1, -1) \cdot i$
$=$ Right coset

→ For $-i \in G$

Left Coset $= -i \cdot (1, -1)$
$= (-i, i)$
$= (1, -1) \cdot -i$
$=$ Right Coset.

$\therefore G/H = \{H, H(i), H(-1), H(-i)$

X	H	-H	Hi	H(-i)
H	H	-H	Hi	H-i
-H	-H	H	-Hi	Hi
Hi	iH	-Hi	H	H
H(-i)	-iH	-Hi	H	H

From Composition table,

1 Closure:

$\forall a, b \in G, \; a \times b \in G, \; H \times H \in G$

2 Associative: $\forall a, b, c \in G$

$$(a \times b) \times c = a \times (b \times c)$$

$$\therefore (H \times H) \times (-H) = H \times (H \times -H)$$

3 Existence of Identity:

H is an Identity element.

4 Inverse: All elements are exists Inverse.

Therefore, all Property is follow then $\in$ it is a Group.

(ii) $G = (S_3, \circ)$ and $H = \{P_1, P_2, P_3\}$

→ Composition table For G

$\circ$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_6	P_5	P_4	P_3
P_3	P_3	P_5	P_1	P_6	P_2	P_4
P_4	P_4	P_6	P_5	P_1	P_3	P_2
P_5	P_5	P_3	P_4	P_2	P_6	P_1
P_6	P_6	P_4	P_2	P_3	P_1	P_5

For Normal Sub Group

Left coset = Right Coset.

- For $P_1 \in H G$

Right coset $= (P_1, P_2, P_3) \cdot P_1$

$= (P_1, P_2, P_3)$ — 1

Left Coset $= P_1 \cdot (P_1, P_2, P_3)$

$= (P_1 \cdot P_1, P_1 \cdot P_2, P_1 \cdot P_3)$

$= (P_1, P_2, P_3)$ — 2

From eqⁿ $1 = 2$

-> For $P_2 \in G$

Right Coset $= (P_1, P_2, P_3) \circ P_2$
$= (P_2, P_1, P_6)$ — 1

Left Coset $= P_2 \cdot (P_1, P_2, P_3)$
$= (P_2, P_1, P_6)$ — 2

$eq^n \; 1 = 2$

-> For $P_3 \in G$

Right Coset $= (P_1, P_2, P_3) \cdot P_3$
$= (P_3, P_6, P_1)$ — 1

Left Coset $= P_3 (P_1, P_2, P_3)$
$= (P_3, P_6, P_1)$ — 2

$eq^n \; 1 = 2$

- For $P_4 \in G$

Right Coset $= (P_1, P_2, P_3) \cdot P_4$
$= (P_4, P_6, P_5)$ — 1

Left Coset $= P_4 \cdot (P_1, P_2, P_3)$
$= (P_4, P_6, P_5)$ — 2

$eq^n \; 1 = 2$

- For $P_5 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_5$
$= (P_5, P_3, P_4)$ — 1

Left coset $= P_5 \cdot (P_1, P_2, P_3)$
$= (P_5, P_3, P_4)$ — 2

- For $P_6 \in G$

Right coset $= (P_1, P_2, P_3) \cdot P_6$
$= (P_6, P_4, P_2)$ — 1

Left coset $= P_6 \cdot (P_1, P_2, P_3)$
$= (P_6, P_4, P_2)$ — 2

Here, ~~G~~ H is be a normal sub group of G

~~$G/H = \{H1, H2, H3, H4, H5, H6\}$~~

$G/H = \{HP_1, HP_2, HP_3, HP_4, HP_5, HP_6\}$

→ Composition table:

$\circ$	HP_1	HP_2	HP_3	HP_4	HP_5	HP_6
HP_1	HP_1	HP_2	HP_3	HP_4	HP_5	HP_6
HP_2	HP_2	HP_3	HP_6	HP_5	HP_4	HP_3
HP_3	HP_3	HP_5	HP_1	HP_6	HP_2	HP_4
HP_4	HP_4	HP_6	HP_5	HP_1	HP_3	HP_2
HP_5	HP_5	HP_3	HP_4	HP_2	HP_6	HP_1
HP_6	HP_6	HP_2	HP_2	HP_3	HP_1	HP_5

From table,

(i) Closuer: From table $HP_i \cdot HP_j \in S_3$

(ii) From table, it is clear that S_3 is associative.

(iii) Identity Element is HP_1.

(iv) Inverse: All the elements inverse are exits.

Thus, G/H is a Group.

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* Task : 6 Homomorphism of Groups

1 Define the following term,

(a) Homomorphism of Groups :

Let $(G, *)$ and ~~G' &~~ $(G', *_1)$ be to groups then A mapping $F : G \to G'$ is said to be a Homomorphism.

if $F(a * b) = F(a) *_1 F(b)$

(b) ~~K~~ Kernel of Homomorphism :

Let $(G, *)$ and $(G', *_1)$ be two groups, A mapping $F : G \to G'$ is a homomorphism then,

$\text{Ker} F = \{a \in G, F(a) = e'\}$

where e' = Identity element of G' Group.

2 Prove each of the following Function is homorphism.

Find Kernel of the Homomorphism

(a) $F : (\mathbb{C}, +) \to (\mathbb{R}, +)$ Defined by
$$F(x + iy) = x$$

Let $a, b \in \mathbb{C}$, $a, b \in \mathbb{R}$

$\therefore F(a + ib) = a$

$\therefore F(a + ib) = F(a + ib)$

Here F is a Homomorphism.

→ Kernel of ~~G~~ Homomorphism.

Here, Identity element of Group R is 0 because given operation is addition.

$\forall, F(x + iy) = x$

$\therefore F(0 + iy) = 0$

$\text{Ker} F = \{0 + iy\}$

(b) $F : (Z_5 - \{0\}, \times_5) \to (Z_4, +_4)$

Defined by $F = \{(1,0), (2,3), (3,1), (4,2)\}$

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Let $Z_5 = \{[1], [2], [3], [4]\}$

$Z_4 = \{[0], [1], [2], [3]\}$

Here, $F(1) = 0$, $F(3) = 1$
$F(2) = 3$ $F(4) = 2$

→ For Homomorphism,

- We take, $[1] \in Z_5$
  $[1] \in Z_4$

$F(1 \times_5 1) = F(1)$
$= 0$ — (1)

$F(1) +_4 F(1) = 0 +_4 0$

$= 0$ — (2)

From eq^n, 1 = 2

$\therefore F(1 \times_5 1) = F(1) +_4 F(1)$

- We take, $[2] \in Z_5$
  $[3] \in Z_4$

$F(2 \times_5 3) = F(1)$
$= 0$ — (1)

$F(2) +_5 F(3) = F(1)$
$= 0$ — (1)

From eq^n 1 = 2

$$f(2 \times_5 3) = f(2) +_4 f(3)$$

Thus, the following Function is Homomorphism

-> Kernel of Homomorphism.

Here, Identity element for Z_4 is 0 beacuse, Here operation is addition.

$$e' = 0$$

Ker f, We have to cheak

$$f(a) = e'$$

$$\therefore f(1) = 0$$

$$\therefore Ker(f) = \{(1)\}$$

(iii) $f : (G = \{\pm 1, \pm i\}, \times) \rightarrow (Z_4, +_4)$

Defined by, $f = \{(1, 0), (-1, 2), (i, 1), (-i, 3)\}$

$G = \{-i, -1, 1, i\}$
$Z_4 = \{[0], [1], [2], [3]\}$

$f(1) = 0 \qquad f(i) = 1$
$f(-1) = 2 \qquad f(-i) = 3$

$\rightarrow$ For Homomorphism,

- We take, $-i \in G$, $[1] \in Z_4$

$f(-i \times 1) = f(-i) = 3$ — ①

$f(-i) +_4 f(1) = 3$ — ②

From eq^n 1 = 2
$f(-i \times 1) = f(-i) +_4 f(1)$

- We take, $-1 \in G$, $[1] \in Z_4$

$f(-1 \times 1) = f(-1) = 2$ — ①

$f(-1) +_4 f(1) = 2$ — ②

From eq^n 1 = 2
$f(-1 \times 1) = f(-1) +_4 f(1)$

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- We take, $1 \in G$, $[1] \in Z_4$

$f(1 \times 1) = f(1) = 0$ — ①

$f(1) +_4 f(1) = 0$ — ②

From eq^n 1 = 2

$\therefore f(1 \times 1) = f(1) +_4 f(1)$

- We take, $i \in G$, $[1] \in Z_4$

$f(i \times 1) = f(i) = 1$ — ①

$f(i) +_4 f(1) = f(i) = 1$ — ②

From eq^n 1 = 2

$\therefore f(i \times 1) = f(i) +_4 f(1)$

Thus, the following Function is Homomorphism.

→ Kernel of Homorphism:

Here, Identity element of Z_4 is 0 because the operation is addition.

$Ker(f) = f(a) = 0$

$\therefore f(1) = 0$

$\therefore \operatorname{Ker}(f) = \{1\}$

* Task 7 : Rings, Integral Domain and Fields.

1 Define the following term with Properties and example.

(a) Ring: A non empty set R is under the binary operation addition and Multiplication, If the following conditions are satisfied:

For $(R, +, \cdot)$ is Ring, If

(1) $(R, +)$ is an abelian Group

(2) $(R, \cdot)$ is a semigroup.

(3) R is follow Distributive Property, $a, b, c \in R$,

$$a.(b+c) = a.b + a.c$$

(b) Intergal Domain: A non empty set R is under the binary operation addition and Multiplication If the following conditions are satisfied:

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For (R, +, ·) is Intergal Domain, If,

1 R is commutative.

2 R has Unit element.

3 R is without zero divisors.

Ex. Ring of integers (Z, +, ·) is an intergal domain,

(c) Field: A Ring containing at least two elements is called a Field if,

(i) It is commutative.

(ii) It has unity.

(iii) It is such that every non-zero element has multiplicative inverse in R.

Ex. A system (R, +, o) is a Field if

(i) (R, +) is an abelian Group.

(ii) $(R', \circ)$ is a commutative Group.

(iii) The distributive laws,

$$a(b+c) = ab + ac \quad , \quad \forall a,b,c \in R.$$

* Task : 8 : Introduction to Boolean Algebra.

Define the following term with example.

1 Boolean Algebra:

Boolean Algebra is the branch of algebra in which the values of the variables are the truth values, it usually denoted by 1 and 0.

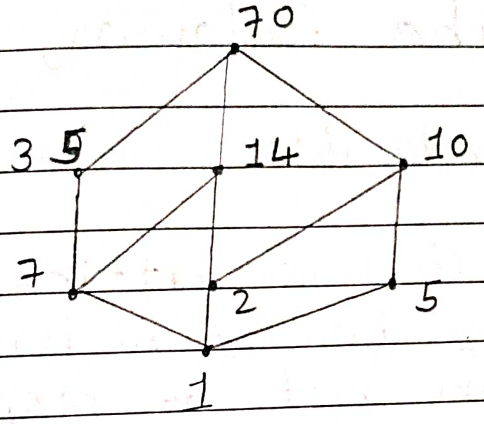
Ex.

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

2 Sub Boolean Algebra:

Consider a boolean algebra $(B, *, +, ', 0, 1)$ and Let $A \subseteq B$ then $(A, *, +, ', 0, 1)$ is called Sub Boolean Algebra of B if A itself is a Boolean algebra. A contains the elements 0 and 1 and is closed under the Operations $*$, $+$ and $'$.

Ex. Consider the Boolean Algebra D_{70}



3 Boolean Ring:

Boolean Ring R is a ring for where $x^2 = x$ for all x in R, that is a ring that consists only of indempotent elements.

An exampelis the ring of integrs modulo 2.

$$r^2 = r \text{ for } \forall r \in R.$$

* Task : 9 : Meet and Join.

1 Define the following terms.

(a) Join-irreducible:

If $(B, *, \oplus)$ is lattice and $a \in B$ is said to be Join-irreducible if it can not be express as a LUB of two distint element of B.

(b) Meet-irreducible:

If $(B, *, \oplus)$ is lattice and $a \in B$ is said to be meet-irreducible if it can not be express as a GLB of two distint element of B.

(c) Atoms:

Let $(B, *, \oplus)$ be a lattice and $a \in B$ then a is said to be Atom if it satisfy following Property.

(i) a is Join-irreducible.

(ii) a is cover of 0.

(d) Anti-atoms:

Let $(B, *, \oplus)$ be a lattice and $a \in B$ then a is said to be anti-atoms if it satisfy following Property.

(i) a is Meet-irreducible.

(ii) a is cover of 1.

2 Prove the following Boolean Identities.

(a) $a \oplus (a' * b) = a \oplus b$

$$\begin{aligned} L.H.S. &= a \oplus (a' * b) \\ &= (a \oplus a') * (a \oplus b) \\ &= 1 * (a \oplus b) \\ &= a \oplus b \end{aligned}$$

L.H.S. = R.H.S.

(b) $a * (a' \oplus b) = a * b$

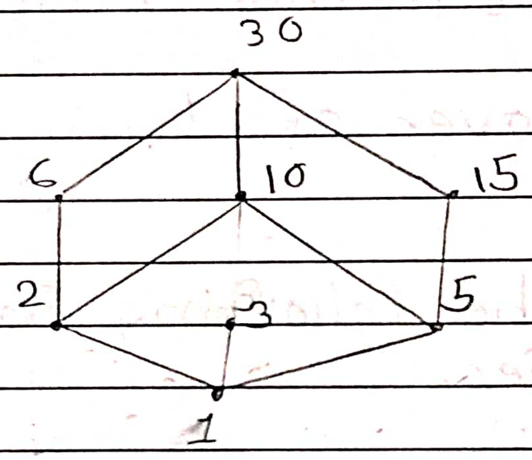
$$\begin{aligned} L.H.S. &= a * (a' \oplus b) \\ &= (a * a') \oplus (a * b) \\ &= 0 \oplus (a * b) \\ &= a * b \end{aligned}$$

L.H.S = R.H.S.

3 Find Join-irreducible, Meet-irreducib[le], atoms and antiatoms of following lattices.

(a) $\langle S_{30}, D \rangle$

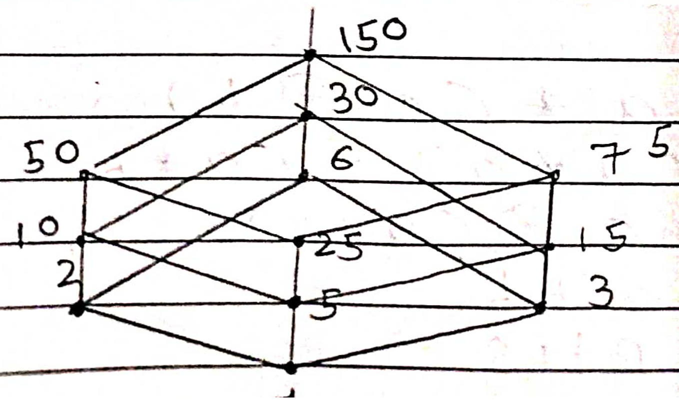
$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



Join-irreducible - 2, 3, 5
Meet-irreducible - 6, 10, 15
Atoms - 2, 3, 5
Anti-atoms - 6, 10, 15

(b) $\langle S_{150}, D \rangle$

$S_{150} = \{1, 2, 3, 5, 10, 6, 15, 25, 50, 30, 75, 150\}$



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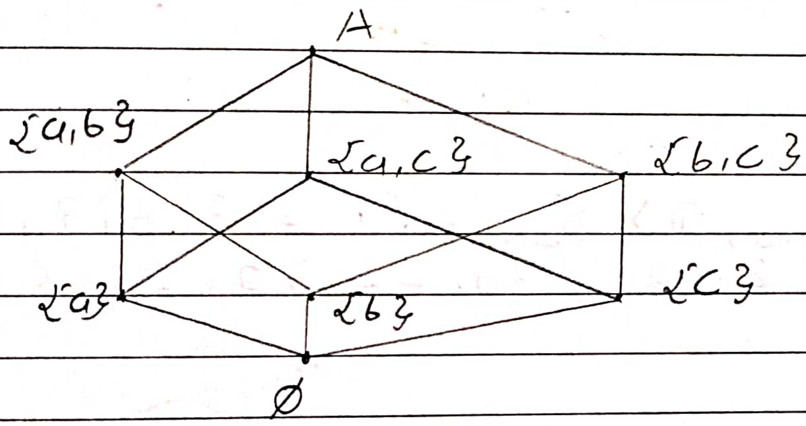
Join-irreducible - 2, 3, 5, 25
Atoms - 2, 3, 5
Meet-irreducible - 6, 30, 50, 75
Anti-atoms - 30, 50, 75

(c) $\langle P(A), \subseteq \rangle$ where $A = \{a, b, c\}$

$P(A) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \emptyset, A\}$



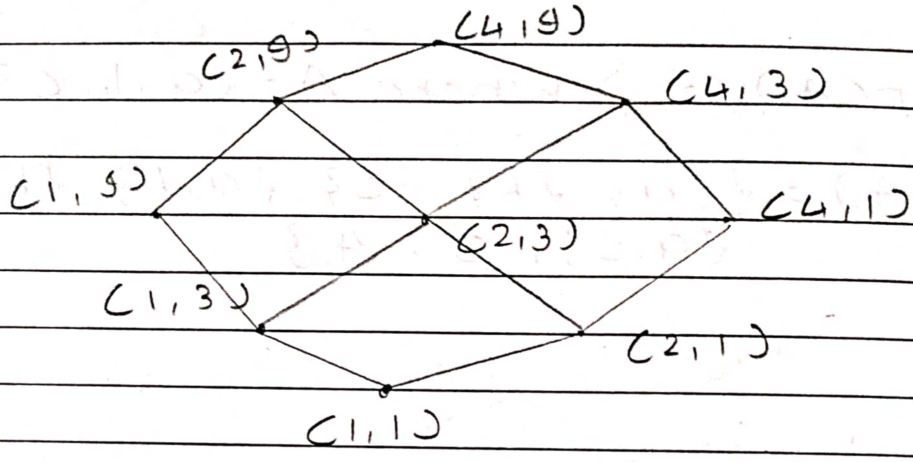
Join-irreducible - $\{a\}, \{b\}, \{c\}$
Atoms - $\{a\}, \{b\}, \{c\}$
Meet-irreducible - $\{a,b\}, \{a,c\}, \{b,c\}$
Anti-atoms - $\{a,b\}, \{a,c\}, \{b,c\}$

(d) $\langle S_4 \times S_9, D \rangle$

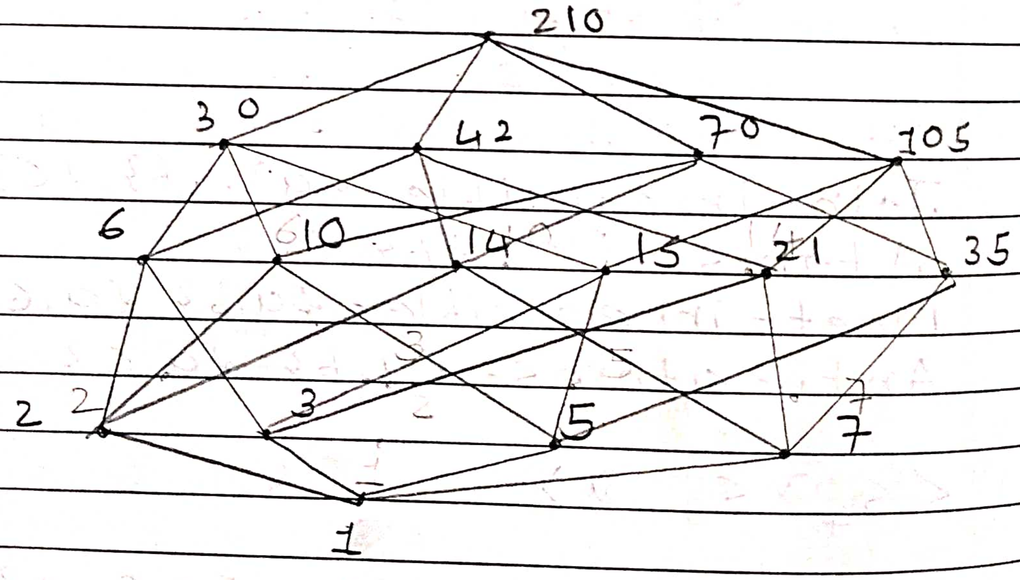
$S_4 = \{1, 2, 4\}$, $S_9 = \{1, 3, 9\}$

$S_4 \times S_9 = \{(1,1), (1,3), (2,1), (4,1), (1,9), (2,3), (2,9), (4,3), (4,9)\}$

Join-irreducible - (1,3), (1,9), (2,1), (4,1)
Meet-irreducible - (2,9), (4,3), (4,1), (1,9)
Atoms - (1,3), (2,1)
Antiatoms - (2,9), (4,3)



(e) $\langle S_{210}, D \rangle$ $S_{210} = \{1, 2, 3, 5, 7, 6, 10, 14, 15, 21, 35, 30, 42, 70, 210\}$



Join-irreducible - 2, 3, 5, 7
Meet-irreducible - 30, 42, 70, 105
Atoms - 2, 3, 5, 7
Antiatoms - 30, 42, 70, 105

* Task : ~~9~~ 10 : Boolean Expression, SOP or POS.

1 Find the SOP and POS Expansions of the following Boolean Function.

(a) $F(x, y, z) = x + y + z$

x	y	z	$x+y+z$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$$F(x, y, z) = \Sigma(1, 2, 3, 4, 5, 6, 7)$$
$$= \Pi(0)$$

(b) $F(x, y, z) = (x + z)y$

x	y	z	$(x+z)y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0

1	1	0	1
1	1	1	1

$F(x, y, z) = \Sigma(3, 6, 7)$
$= \Pi(0, 1, 2, 4, 5)$

(c) $F(x, y, z) = x + (yz)'$

x	y	z	$(yz)'$	$x + (yz)'$
0	0	0	1	1
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

$F(x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 7)$
$= \Pi(3)$

(d) $F(x, y, z) = (x + y)' + (x'z)$

x	y	z	$(x+y)'$	$(x'z)$	$(x+y)' + (x'z)$
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	1	0	1	1
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	0	0	0
					0

1	1	1	0	0	0

$F(x, y, z) = \Sigma(0, 1, 3)$
$= \Pi(2, 4, 5, 6, 7)$

(e) $F(x, y, z, w) = (xy') + w$

x	y	z	w	xy'	$xy' + w$
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	0	0
0	1	1	1	0	1
1	0	0	0	1	1
1	0	0	1	1	1
1	0	1	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	0	1	0	1
1	1	1	0	0	0
1	1	1	1	0	1

$F(x, y, z, w) = \Sigma(1, 3, 5, 7, 8, 9, 10, 11, 13, 15)$

$= \Pi(0, 2, 4, 6, 12, 14)$

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* Task: 11: Boolean Expression and Equivalence.

1 In any Boolean algebra, show that,

(a) $a = b \iff ab' + a'b = 0$

Let $a = b = c$

$$
\begin{aligned}
L.H.S. &= ab' + a'b \\
&= cc' + c'c \\
&= 0
\end{aligned}
$$

L.H.S. = R.H.S.

(b) $a = 0 \iff ab' + a'b = b$

$\Rightarrow$
$$
\begin{aligned}
L.H.S. &= ab' + a'b \\
&= 0 + b \\
&= b
\end{aligned}
$$

L.H.S. = R.H.S

$\Rightarrow$ $ab' + a'b = b$

If $a = 1$ then $b' + 0 = b$ not possible

If $a = 0$ then $0 + b = b$ Possible.

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(c) $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

$\Rightarrow$ L.H.S $= (a+b')(b+c')(c+a')$

$= (ab + ac' + bb' + b'c').(c+a')$

$= abc + ac'c + bb'c + aa'b + aa'c + bb'a' + b'c'a'$

$= abc + b'a'c'$

$= \cancel{R.H.S}$

$\Rightarrow$ R.H.S. $= (a'+b)(b'+c)(c'+a)$

$= (a'b' + a'c + bb' + bc)(c'+a)$

$= a'b'c' + a'cc' + bb'c' + bcc' + aa'b + aa'c + bb'a + abc$

$= abc + a'b'c'$

$\therefore$ L.H.S $=$ R.H.S.

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2 Simplify the following Boolean Expressions.

(a) $(a * b') \oplus (a \oplus b)'$

$= (a' \oplus b') \oplus (a' * b')$
$= ((a' \oplus b') \oplus a') * ((a' \oplus b') \oplus b')$
$= ((a' \oplus a') \oplus b') * (a' \oplus (b' \oplus b'))$
$= (a' \oplus b') * (a' \oplus b')$
$= a' \oplus b'$

(b) $(a' * b' * c) \oplus (a * b' * c) \oplus (a * b' * c')$

$= \{(a' \oplus a) * (b' \oplus a) * (c \oplus a)\} *$

$\{(a' \oplus b') * (b' \oplus b') * (c \oplus b')\} *$

$\{(a' \oplus c) * (b' \oplus c) * (c \circledast c)\}$

$\oplus (a * b' * c')$

$= (b' \oplus a) * (c \oplus a) * (a' \oplus b')$
$* b' * (c \oplus b)$