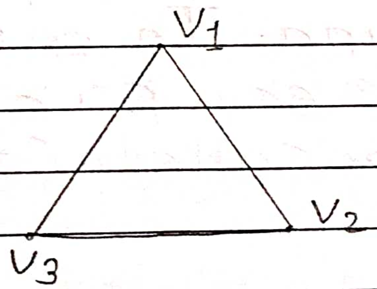
# Unit -5 Graph Theory

## * Task : 1 : Basic Concepts of Graphs and Diagraphs.

1 Define the following terms of the Graph with example.

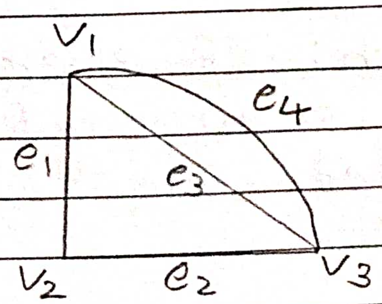
(i) Simple Graph: A Graph which has neither loop and multiple edges is called Simple Graph.

Ex.



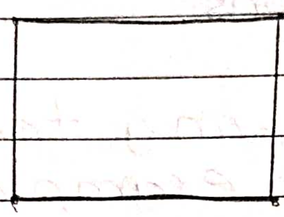

(ii) Multiple Graph: A Graph in which no loop are allowed but more than one edges are join two vertex.

Ex.



(iii) Regular Graph: In a Graph, every vertex has same degree and same edges this Graph is called Regular Graph.

Ex.

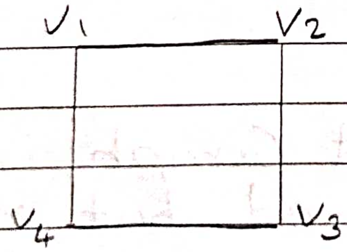


2-Regular Graph

(iv) Null Graph: In a Graph, edges set is empty then this Graph is said to be null Graph.

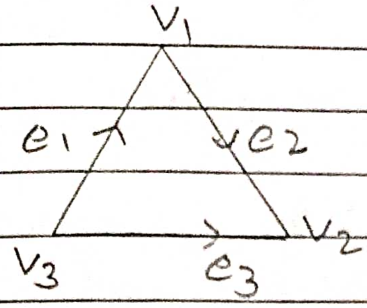
(v) Complete Graph: In a Graph, edeges have every pair of vertex, then graph is called Complete Graph.

Ex.



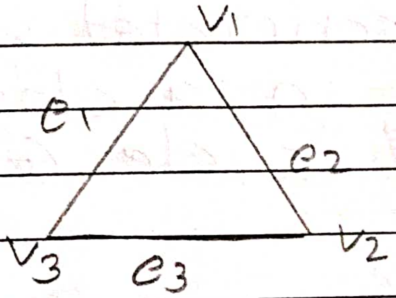

(vi) Directed Graph: If every edges of Graph G is directed then, Graph is called Directed Graph.

Ex.



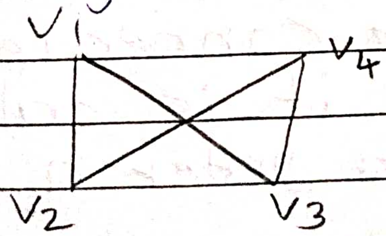

(vii) Undirected Graph: If every edges of Graph G is not directed then this Graph is called Undirected Graph.

Ex.



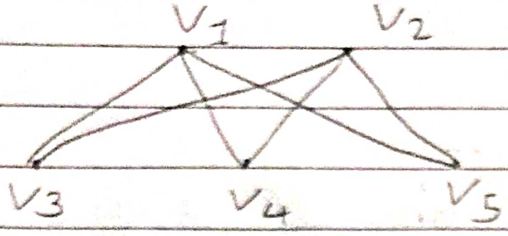

(viii) Bipartite Graph: A set of graph vertices decomposed into two disjoin set such that no graph vertices within the same set are adjacent.

Ex.



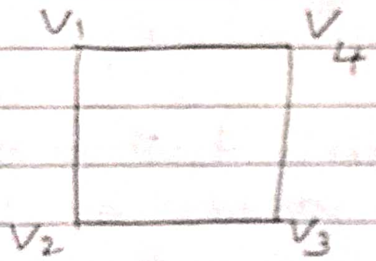

(ix) Complete Bipartite Graph: In a Bipartite Graph in which every vertex $v_1$ and is joining each vertex $v_2$ by one edges.

Ex. $K_{2,3}$




(x) Cycle Graph: In a Graph, If every edges connected with a every vertex in a closed chain This is called cycle Graph.

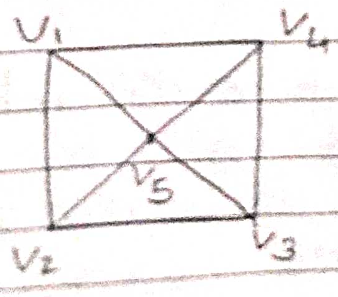
Ex.




$C_4$

(xi) Wheel Graph: In a cycle Graph, every vertex is connected with a new vertex it is called wheel Graph of order n.

Ex.




$W_5$

(xiii) Pendant Vertex: A vertex with degree with one is called Pendant vertex

Ex. 

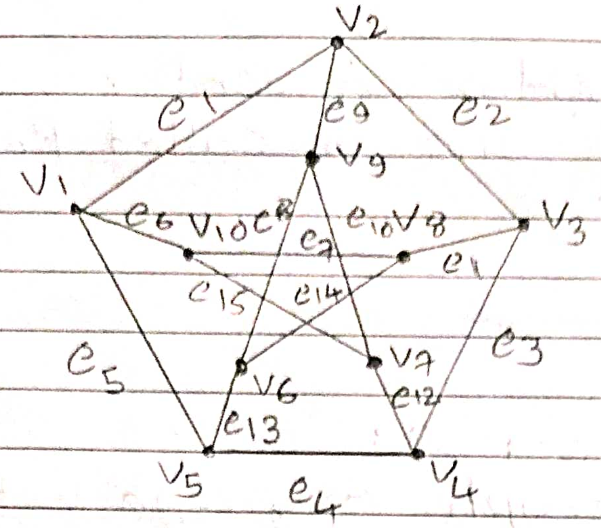

(xiii) Labelled Graph: A graph with each node labelled differently, So that all nodes are considered distinct for purposes.

2 State Handshaking lemma and verify if for the graph.

In any undirected Graph G, number of vertex V and number of edges are E then,

$$\sum_{i=\cancel{0}1}^{n} d(V_i) = 2e$$

(a)




degree $(V_1) = 3$
degree $(V_2) = 3$
degree $(V_3) = 3$
degree $(V_4) = 3$
degree $(V_5) = 3$
degree $(V_6) = 3$
degree $(V_7) = 3$
degree $(V_8) = 3$
degree $(V_9) = 3$
degree $(V_{10}) = 3$

$$\sum_{i=1}^{10} d(V_{10}) = 30 \quad \text{— (1)}$$

By them,

$$\sum_{i=1}^{10} d(V_{10}) = 2e$$

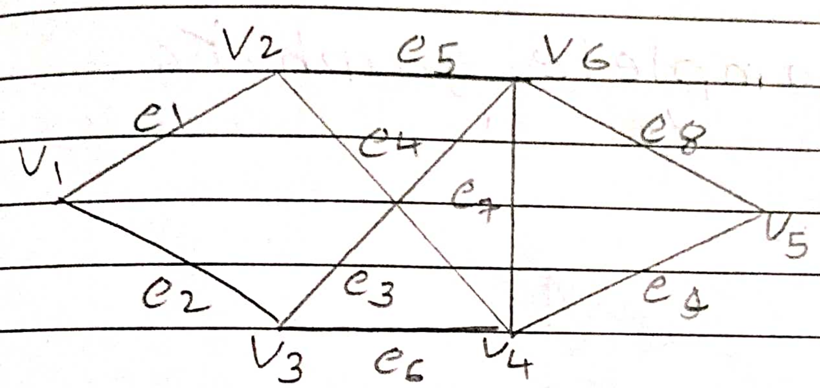
$$= 2 \times 15$$

$$= 30 \quad \text{— (2)}$$

$eq^n \; 1 = 2$

So, Hand Shaking ~~Iem~~ Lemma is prove.

(b)




degree $(V_1) = 2$
degree $(V_2) = 3$
degree $(V_3) = 3$
degree $(V_4) = \cancel{3}\;4$
degree $(V_5) = 2$
degree $(V_6) = 4$

$$\sum_{i=1}^{6} d(V_6) = 18 \quad - (1)$$

By thm,

$$\sum_{i=1}^{6} d(V_6) = 2e$$

$$= 2 \times 9$$

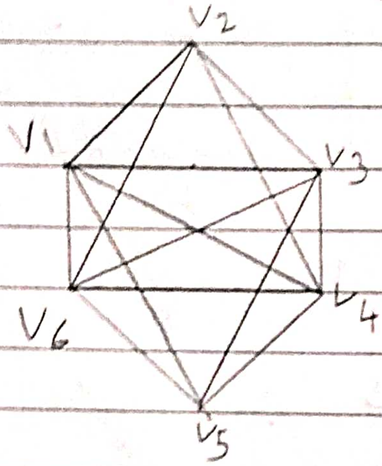
$$= 18 \quad - (2)$$

$eq^n$ 1 = 2.

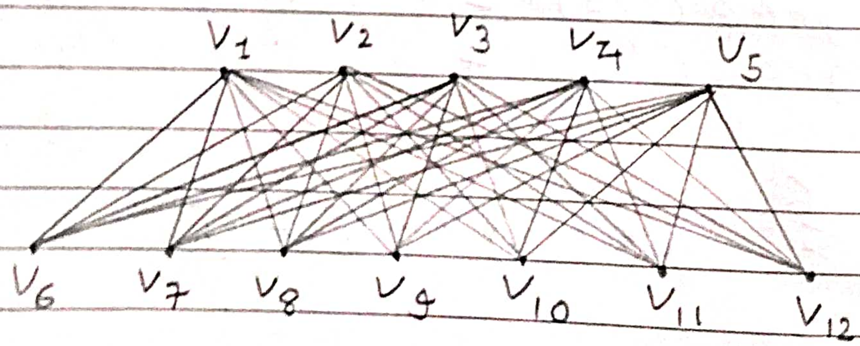
So, Handshaking lemma is prove.

3 Draw the following Graph:

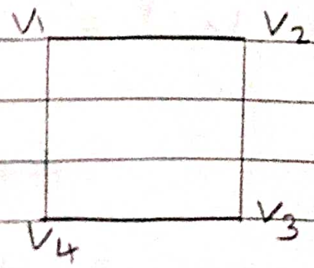
(i) The complete graph $K_6$



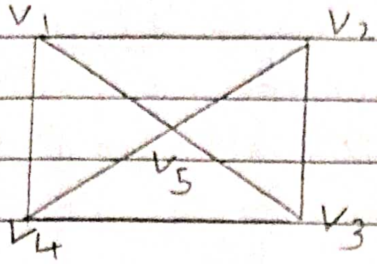

(ii) Complete Bipartite Graph $K_{5,7}$




(iii) Cycle Graph $C_4$:

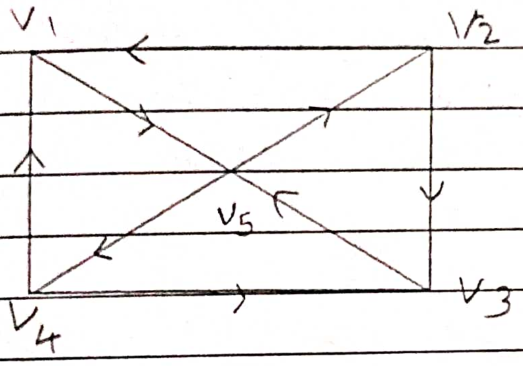


(iv) Wheel Graph $W_5$ :



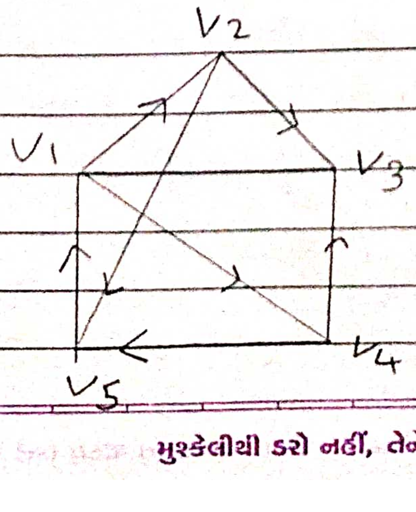

4 Find Out degree and in degree.

(a)




| Vertex | In-degree | Out-degree |
|---|---|---|
| $V_1$ | 2 | 1 |
| $V_2$ | 1 | 2 |
| $V_3$ | 2 | 1 |
| $V_4$ | 1 | 2 |
| $V_5$ | 2 | 2 |

(b)



| Vertex | In-degree | Out-degree |
| --- | --- | --- |
| $V_1$ | 1 | 2 |
| $V_2$ | 1 | 2 |
| $V_3$ | 2 | 0 |
| $V_4$ | 1 | 2 |
| $V_5$ | 2 | 1 |

# * Task : 5 : Matrix Reprsentation of Graphs and Diagraphs.

1 Define the following terms of undirected and directed graph with example.

## (i) Adjacency Matrix :

→ Undirected Graph :

Let $G = (V, E)$ be the graph, where $V = \{V_1, V_2, ---\}$ be the vertex set then adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$

$$a_{ij} = \begin{cases} n & \text{- If } V_i \text{ and } V_j \text{ is connected.} \\ 0 & \text{- If } V_i \text{ and } V_j \text{ not connected.} \end{cases}$$

→ Directed Graph :

Let $G = (V, E)$ be the graph, where $V = \{V_1, V_2, ---\}$ be the vertex set then adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$

$$a_{ij} = \begin{cases} n = \text{If } V_i \text{ from to } V_j \\ 0 = \text{If } V_i \text{ is not from } V_j \end{cases}$$

(ii) Path Matrix: Let $G=(V,E)$ is any graph and $V_1$ and $V_2$ any two vertex of graph $G$, if there are p path between vertices $V_1$ and $V_2$ then it is called path matrix.

$$P_{ij} = \begin{cases} 1 & \text{if edges } e_j \text{ in } \cancel{\text{va}} \text{ path} \\ 0 & \text{if edges } e_j \text{ is not in path} \end{cases}$$

(iii) Incident Matrix:

→ Undirected Graph:

Let $G=(V,E)$ be the graph where $V=\{V_1, V_2, ---\}$ be a vertex set and edge set then incident matrix is $n \times n$ Matrix $A=[a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 - & \text{If the } V_i \text{ is incident of } e_j \\ 0 - & \text{Otherwise} \end{cases}$$

→ Directed Graph:

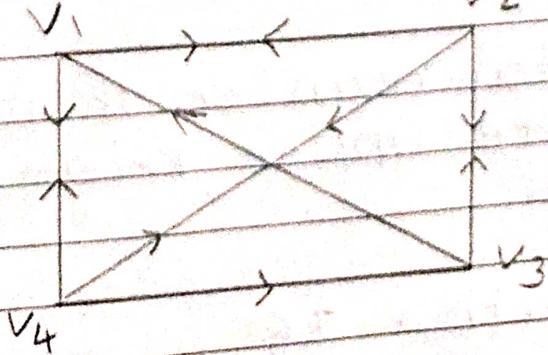
Let $G = (V, E)$ be the graph where $V = \{V_1, V_2, \dots\}$ be a vertex set and edges set then incident matrix $A = [a_{ij}]$ is defined by.

$$a_{ij} = \begin{cases} 1 & \text{- If } V_i \text{ is initial of } \cancel{\text{vertc}} \text{ edges } e_j \\ -1 & \text{- If } V_i \text{ is end of edges } e_j \\ 0 & \text{- Not incident.} \end{cases}$$

2 Let the adjacency matrix of a graph $G = (V, E)$ be

$$A = \begin{array}{c|cccc|} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 0 & 1 \\ V_2 & 1 & 0 & 1 & 1 \\ V_3 & 1 & 1 & 0 & 0 \\ V_4 & 1 & 1 & 1 & 0 \end{array}$$

~~(a)~~ Out



(a) Out degree of $V_2$

Out degree of $V_2 = 3$

(b) The number of paths from $V_1$ to $V_2$

1) $V_1, V_2, V_3$
2) $V_1, V_2, V_3, V_4$
3) $V_1, V_2, V_4$

(c) Total Number of paths of length 2

For total number of paths,

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

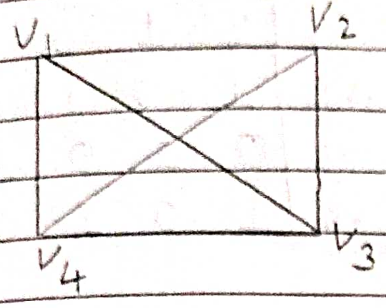
$$= \begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \end{bmatrix}$$

For total number of paths calculate upper triangle of matrix.

total path = 16

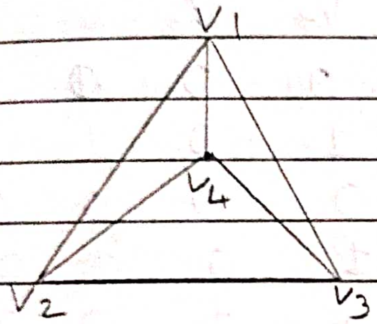
3 Construct Adjacency Matrix of all graphs shown in Task : 2

(a)




$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

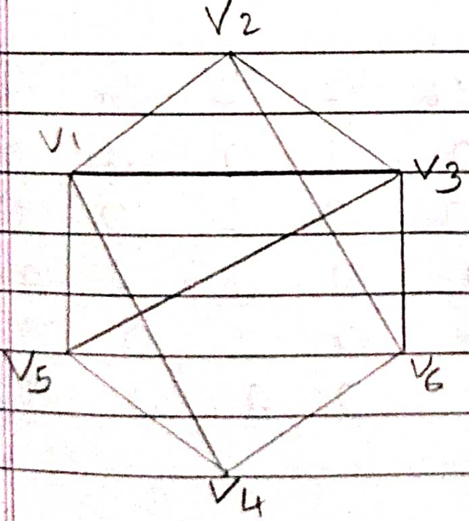
(b)




$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

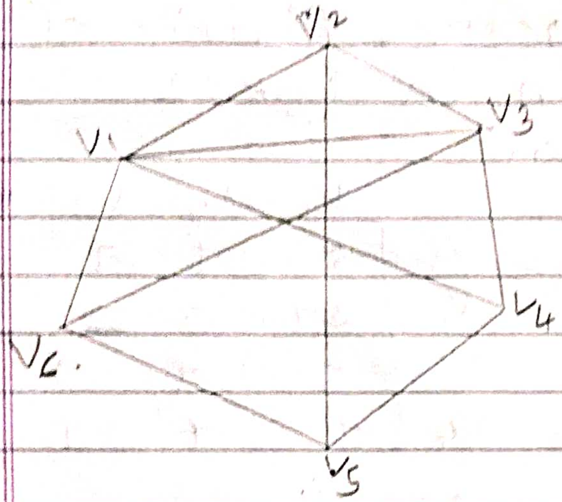
*

(c)



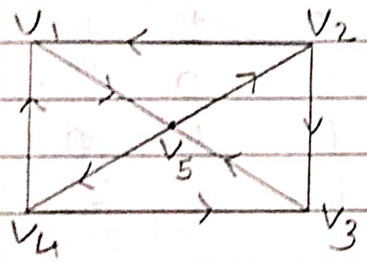

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(d)



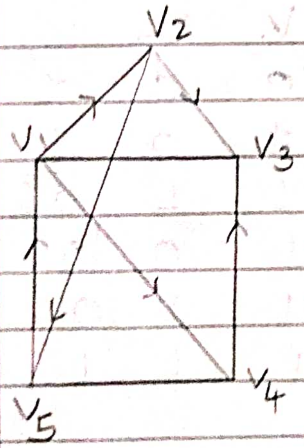

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(e)



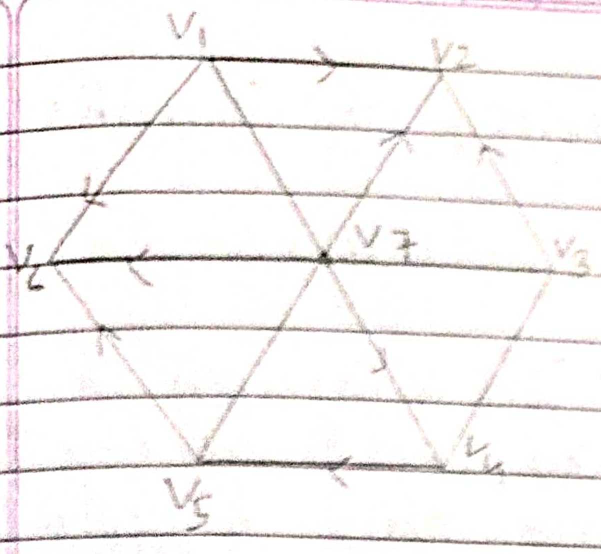

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(f)



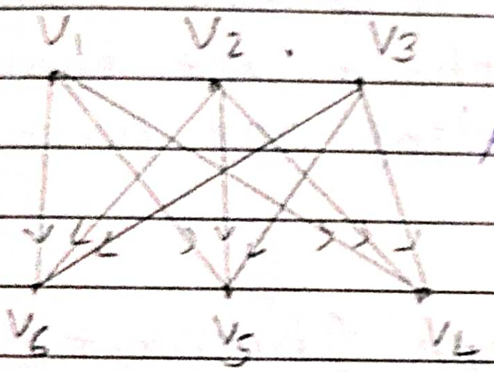

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(g)




$$A = \begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_7 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

(h)




$$A = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 0 & 1 & 1 & 1 \\ v_3 & 0 & 0 & 0 & 1 & 1 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

# * Task : 6 : Coloring of Graphs:

1 Define Following terms:

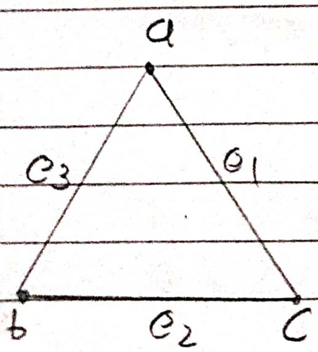
(i) Vertex Coloring: In a Graph G, there are no adjsent vertex color with same, this is called vertex coloring.

(ii) Edge Coloring: In a edge coloring. in Graph G, there are two adjsent vertex does not have same color.

(iii) Chromatic Number: Chromatic number is a minimum number of color that required to color of graph.

2 Coloring Following Graphs:

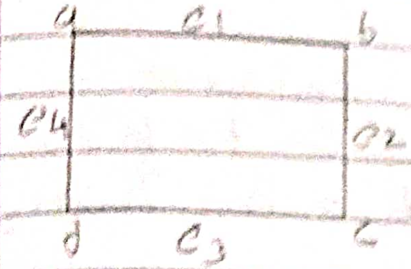
(a)




For vertex, a = red
b = green
c = yellow.

For edges $e_1$ = Blue
$e_2$ = Pink
$e_3$ = Orange

(b)

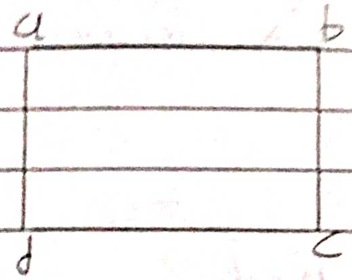



For Vertex - a, c - Red
b, d - Green

For edges - $e_1, e_3$ - Yellow
$e_4, e_2$ = Orange.

3 Find the Chromatic number in the Graph.

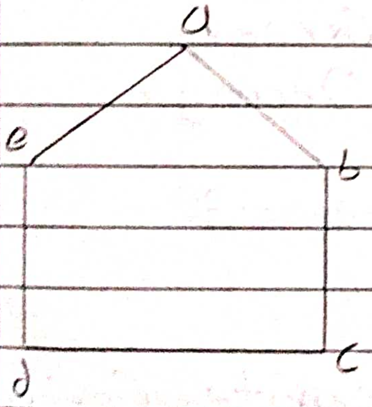
(i)




For Vertex - a, c - Red
b, d - Green

Chromatic number = 2

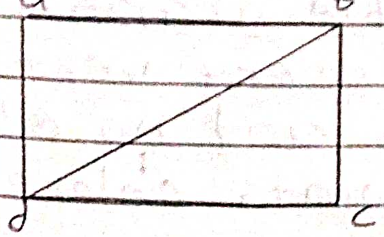
(ii)




For Vertex - a, c - Red
e, b - Green
d - Orange

Chromatic number = 3

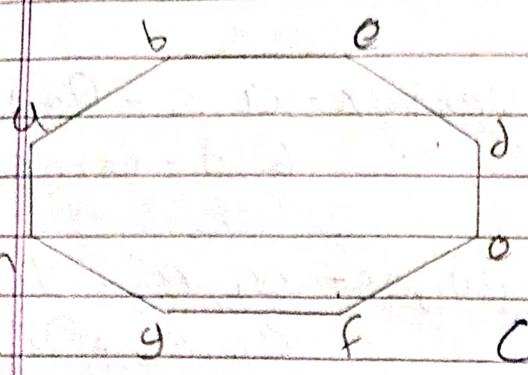
(iv)




For Vertex - a, c - Red
b - Green
d - Orange

Chromatic number = 3

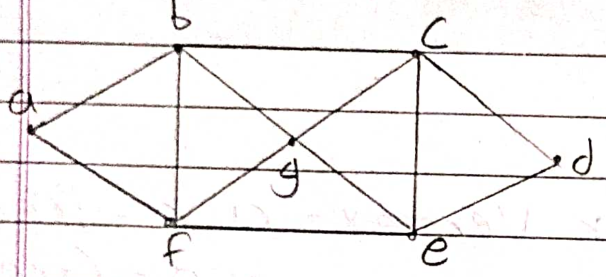
(iii)




For Vertex:
a, e, c, g - Red
b, d, f, h - Orange

Chromatic Number = 2

(v)




~~For Vertex:~~
~~a, c - Red~~
~~f, d - Green~~
~~g,~~

For Vertex: g, a, d - Red
e, b - Green
c, a - Orange

Chromatic Number = 3

4 Prove that every Biparite Graph is 2 colorable:

Let G be a 2 colorable graph, which means we can color every vertex into 2 color exactly and no edges will have both and points colored the same color.

Let a denote the subset of vertices coloured green and Let b donoted the subset of vertices coloured Red.

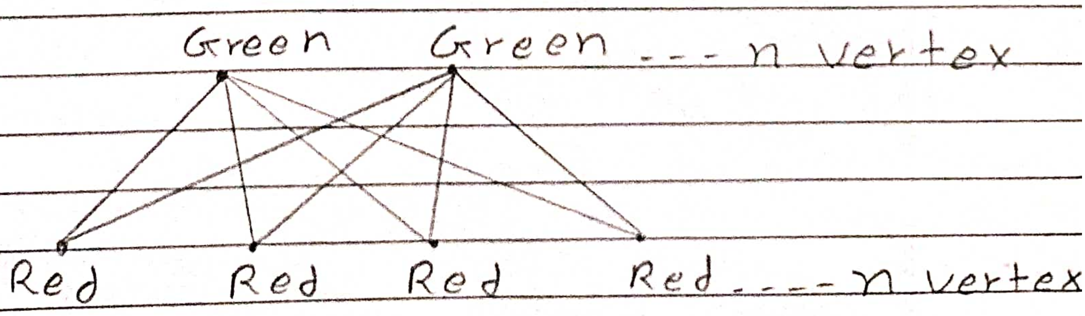
Since all vertices a are green, there are no edges within a and same for b. this implies that every edge has one end point in a and other in b which means G is biparite.

Conversely, Suppose G is Biparite, we can partition into two subsets $V_1, V_2$. So each edge has one end point in $V_1$ and the other in $V_2$.

Then coloring every vertex of $V_1$ as Green and that $V_2$ as blue colour.

Hence, every Biparite Graph is 2 - colorable.

Ex.



* Task : 3 : Walk, Path and Circuits of Graphs and Diagraphs.

1 Define following terms for Undirected Graph.

(a) Walk : Walk is a finiter ultar nativite sequece of vertex and begine and end with vertex.

(b) Path : In a walk not vertex and not edges are repeted, may be starting vertex are repeted it is called Path.

(c) Circuit : In a trial begining and ending are same this is called circuit.

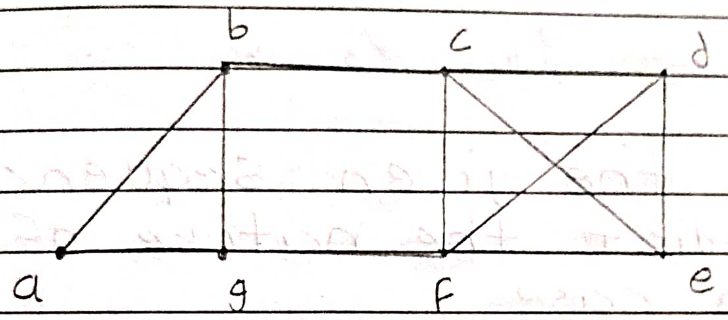
(d) Length of Path : Number of edges appere in the path, this is called Length of Path.

(e) Trail : In a walk edges are not repeted there we can call it is trail.

(f) Euler Graph : If connected graph G is called euler graph, if there is a closed path trial which includes every edges of the graph.

(g) Hamiltonian Path: A Hamiltonian path is a path that vists each vertex of the graph only one time.

2 Consider the following Path.




Decide which of the following is walks, circuit, a path, a cycle or a trail.

(a) a, b, g, f, c, b - Trail

(b) b, g, f, c, b, g, a - Walk

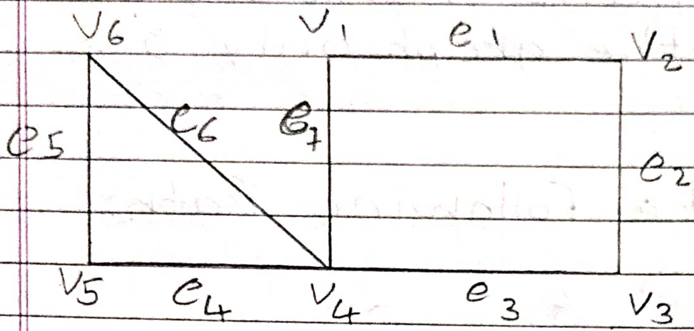
(c) c, e, f, c - Cycle

(d) c, e, f, c, e - Walk

(e) a, b, f, a - Not a Walk

(f) f, d, e, c, b - Path

3 Consider the following graph:




Observe the given sequences and predict the nature of walk in each case.

(a) V1 e1 V2 e2 V3 e2 V2 - Open Walk

(b) V4 e7 V1 e1 V2 e2 V3 e3 V4 e4 V5 - Trail

(c) V1 e1 V2 e2 V3 e3 V4 e4 V5 - Path

(d) V1 e1 V2 e2 V3 e3 V4 e7 V1 - Cycle

(e) V6 e5 V5 e4 V4 e3 V3 e2 V2 e1 V1 e7 V4 e6 V6

~~Path~~ Circuit

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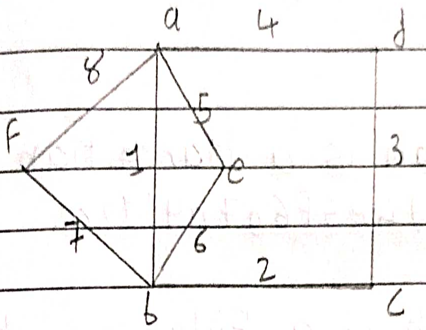
4 Which of the following graphs are Eulerian? If they are eulerian, mention Euler circuit.

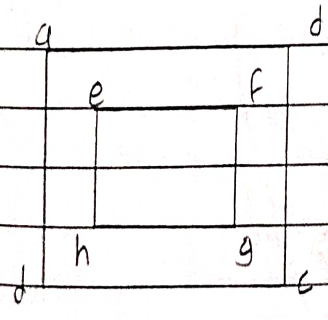
Also which of them is Hermitian? If they are Hermitian, mention hermitian circuit.

(a)




This Graph is Euler Graph as
1 b 2 c 3 d 4 a 5 e 6 b 7 F

(b)




Here, This graph is not Euler or Hermitian graph.

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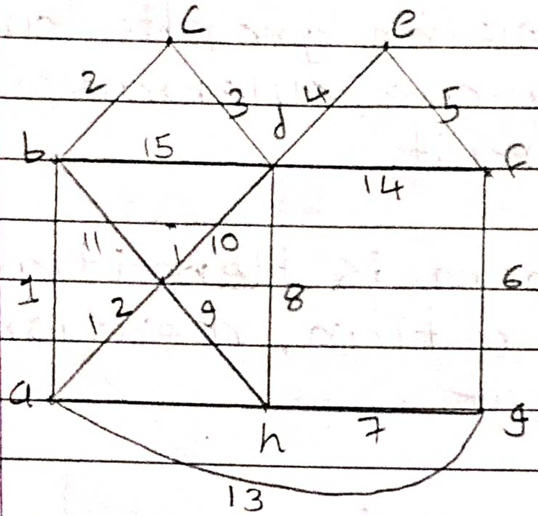
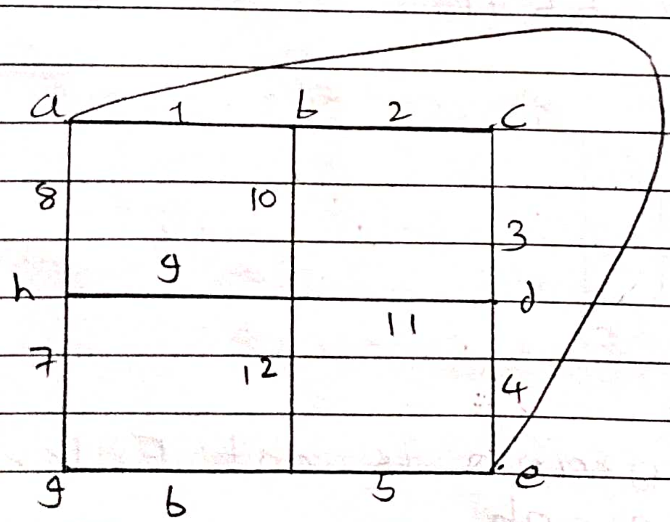
(c)




This graph is a Harmitian graph as 1b2c3d4e5f6g7h9i12a.

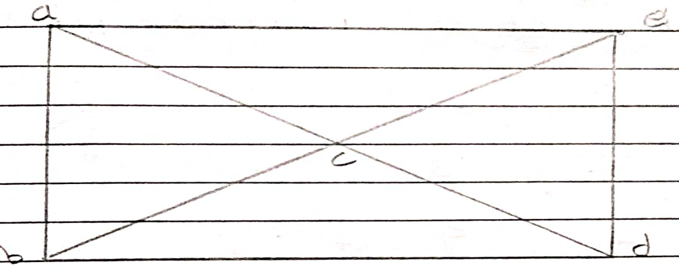
This graph is a Euler graph as f14d15b2c3de5f6g7hrd10i12a1611i9h16a13g.

(d)

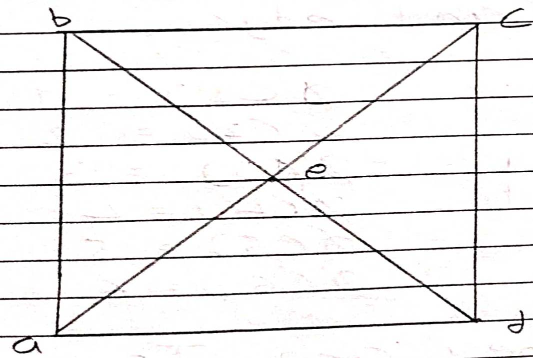



This graph is Harmitian graph as 1b2c3d11i9h7g6f5e.

5 Give an example of a graph which is Eulerian but ~~be~~ not hemitonian.



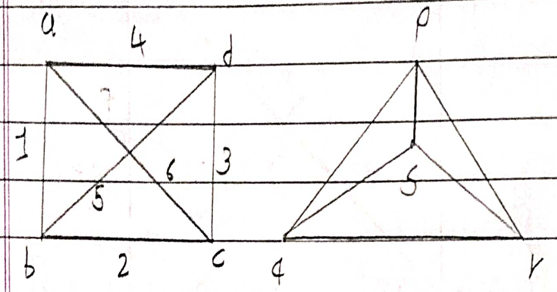

6 Give one example of a graph which is hemitonian but not Eulerian.

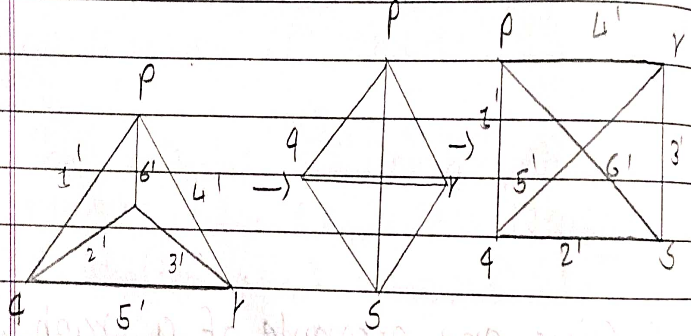


## * Task: 2: Isomorphism of Graphs and Diagraphs.

1 Prove that following pair of graphs are isomorphic.

(a)



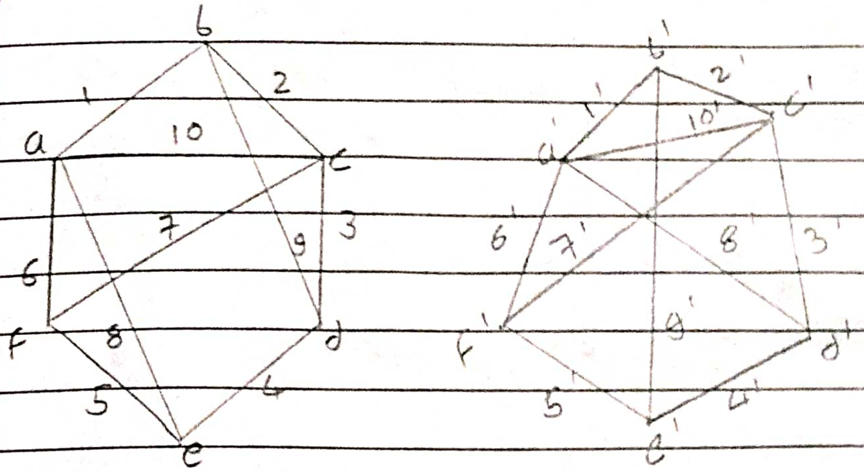


They have same number of vertex and edges.

| | |
|---|---|
| $\phi(p) = a$ | $\phi'(1') = 1$ |
| $\phi(q) = b$ | $\phi'(2') = 2$ |
| $\phi(r) = d$ | $\phi'(3') = 3$ |
| $\phi(s) = e$ | $\phi'(4') = 4$ |
| | $\phi'(5') = 5$ |
| | $\phi'(6') = 6$ |

Thus they are Isomorphic.

(b)




→ $\partial(a) = 4$ $\partial(a') = 4$

$\partial(b) = 3$ $\partial(b') = 3$

$\partial(c) = 3$ $\partial(c') = 3$

$\partial(d) = 3$ $\partial(d') = 3$

$\partial(e) = 4$ $\partial(e') = 4$

$\partial(f) = 3$ $\partial(f') = 3$

Both graph has same number of vertex and edges.

$\phi(a) = a'$ $\phi(1) = 1'$

$\phi(b) = b'$ $\phi(2) = 2'$

$\phi(c) = c'$ $\phi(3) = 3'$

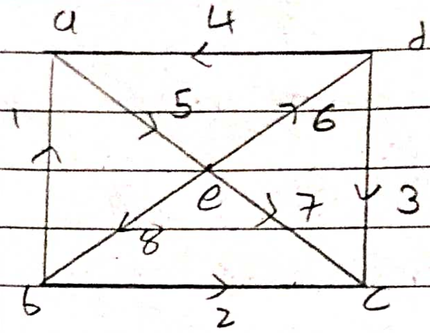
$\phi(d) = d'$ $\phi(4) = 4'$

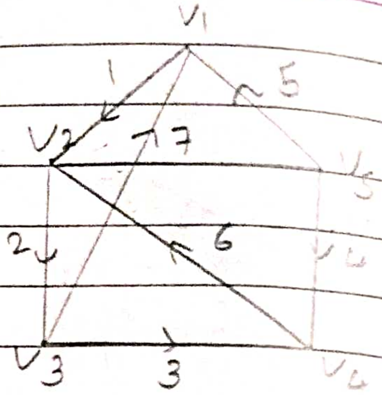
$\phi(e) = e'$ $\phi(5) = 5'$

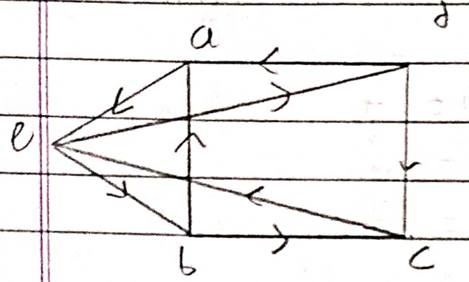
$\phi(f) = f'$ $\phi(6) = 6'$

Here, both graphs are isomorphic.

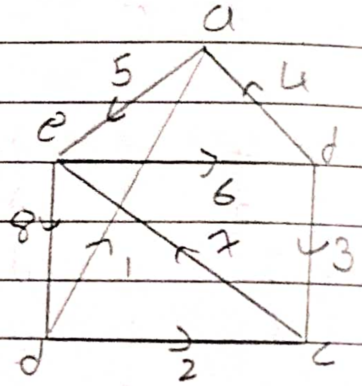
(c)




→



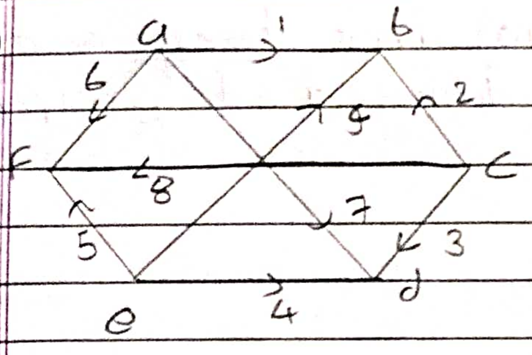

| | |
|---|---|
| $\partial(a) = 1$ | $\partial(v_1) = 1$ |
| $\partial(b) = 2$ | $\partial(v_2) = 2$ |
| $\partial(c) = 1$ | $\partial(v_3) = 1$ |
| $\partial(d) = 2$ | $\partial(v_4) = 2$ |
| $\partial(e) = 2$ | $\partial(v_5) = 2$ |

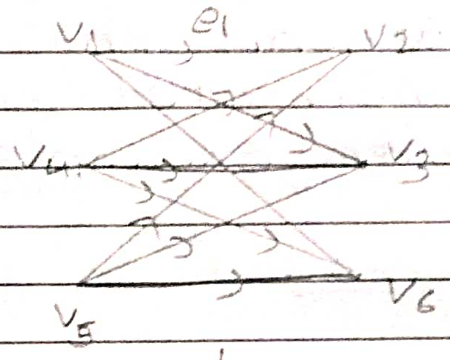
Both Graph have same number of vertex and edges.

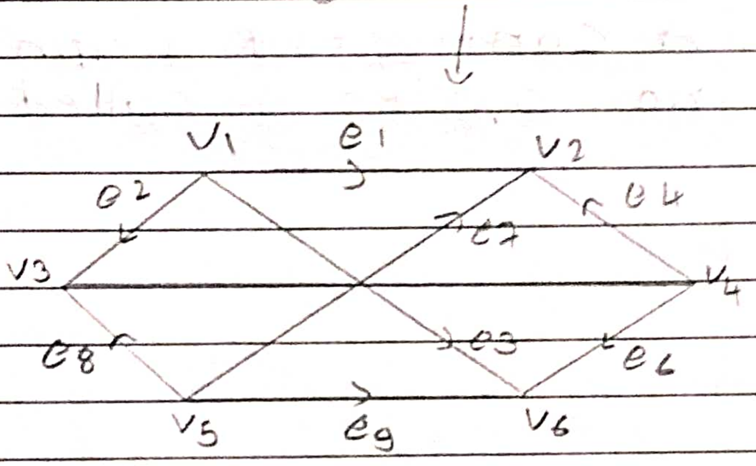
| | |
|---|---|
| $\phi(a) = v_1$ | $\phi'(1) = 5$ |
| $\phi(b) = v_3$ | $\phi'(2) = 8$ |
| $\phi(c) = v_4$ | $\phi'(3) = 2$ |
| $\phi(d) = v_5$ | $\phi'(4) = 4$ |
| $\phi(e) = v_2$ | $\phi'(5) = 4$ |
| | $\phi'(6) = 2$ |
| | $\phi'(7) = 1$ |
| | $\phi'(8) = 6$ |

Hence, Both Graphs are isomorphic.

(d)




There are same number of vertex and edges in graph.

$\phi(a) = v_1$ $\phi'(1) = e_1$
$\phi(b) = v_2$ $\phi'(2) = e_4$
$\phi(c) = v_4$ $\phi'(3) = e_6$
$\phi(d) = v_6$ $\phi'(4) = e_9$
$\phi(e) = v_5$ $\phi'(5) = e_8$
$\phi(f) = v_3$ $\phi'(6) = e_2$
$\phi'(7) = e_3$
$\phi'(8) = e_5$
$\phi'(9) = e_7$

Hence, both Graphs are isomorphic.

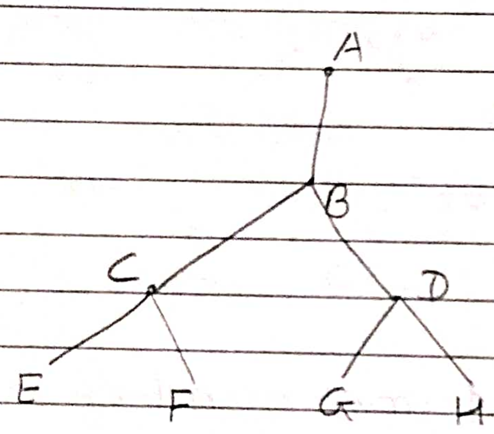
## * Task - 7 : Tree.

1 Define the following terms of undirected and directed graph with example.

(a) Tree :

A connected graph that contains no cycle is called tree.

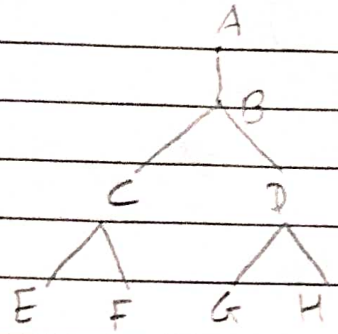
Ex.




(b) Binary Tree :

A binary tree is a tree-like structure that is rooted and in which each vertex has at most two children.

Ex.




(c) Terminal vertex:

In a directed tree any vertex ~~wich~~ which has out degree zero is called a terminal vertex.

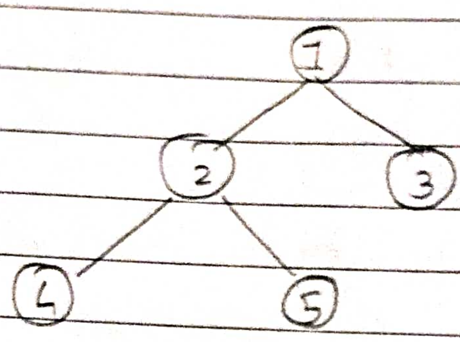
- Internal vertex:

In a tree, vertex that have children are called internal vertex.

(d) height of tree:

The height of a tree is defined as the maximum number of nodes in a branch of a tree.

Ex.




Height of tree = 3