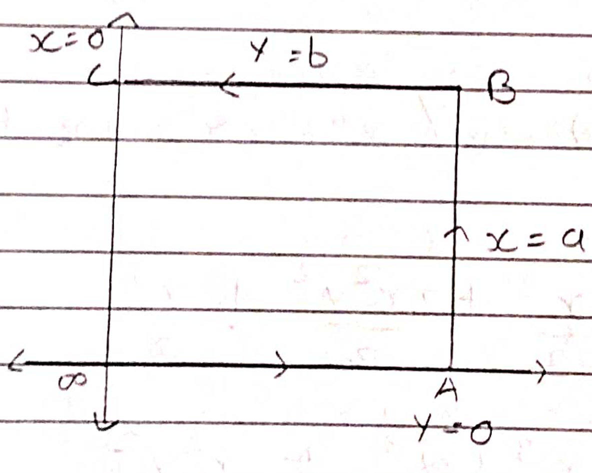
Unit - 7 Vector Integral Calculus

* Task : 1 Line Integral and Green's Theorem.

1 Evaluate $\int_C \bar{F} \cdot d\bar{r}$, where $\bar{F} = (x^2 + y^2)i - 2xy\, j$ and C is the rectangle in xy plane bounded by $y=0$, $x=a$, $y=b$ and $x=0$.



Let, the position vector

$$\bar{r} = x\hat{i} + y\hat{j}$$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

$\rightarrow$ $\bar{F} \cdot d\bar{r} = [(x^2 + y^2)i - 2xy\, j] \cdot [dx\,\hat{i} + dy\,\hat{j}]$

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→ Line Integral.

$\therefore \oint_{C} \vec{F} \cdot d\vec{r} = \int_{OA} x^2 + y^2 \cdot dx + \int_{AB} -2xy \cdot dy$

$+ \int_{BC} (x^2 + y^2) dx + \int_{OC} (-2xy) dy$ — (1)

⇒ For OA ⇒ $y = 0 \to dy = 0$
$x = 0$ to $x = a$

$= \int_0^a (x^2 + 0) \cdot dx \quad (\because y = 0)$

$= \left[\frac{x^3}{3} \right]_0^a$

$= \frac{a^3}{3}$ — (2)

⇒ For AB: ⇒ $x = 0 \to dx = 0$
$y = 0$ to $y = b$

$= -\int_0^b 2ay \cdot dy$

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$$= -2a\left[\frac{y^2}{2}\right]_0^b$$

$$= -b^2 a \quad — \quad (3)$$

$\Rightarrow$ along BC : $y = b \Rightarrow dy = 0$

$x = a$ to $x = 0$

$$= \int_a^0 x^2 + b^2 \cdot dx$$

$$= -\left[\frac{a^3}{3} + b^2 a\right] \quad — \quad (4)$$

$\Rightarrow$ along CO : $x = 0 \rightarrow dx = 0$

$y = b$ to $y = 0$

$$= -\int_b^0 2xy \cdot dy$$

$$= 0 \quad — \quad (5)$$

$\Rightarrow$ put all the value in eq^n - (1)

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} - b^2 a - b^2 a - \frac{a^3}{3}$$

$$= -2ab^2$$

2 Find the work done by the force $\bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ over the curve $\bar{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \leq t \leq 1$. From $(0, 0, 0)$ to $(1, 1, 1)$

$\Rightarrow$ Let, the position Vector,

$$\bar{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

So that, $x = t$, $y = t^2$, $z = t^3$

$$\Rightarrow \frac{d\bar{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\Rightarrow \bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$$

$$= 3t^2\hat{i} + (2t^4 - t^2)\hat{j} + t^3\hat{k}$$

$$\Rightarrow \oint F \cdot \frac{d\bar{r}}{dt} = \int_0^1 (3t^2\hat{i}) + (2t^4 - t^2)\hat{j} + (t^3)\hat{k} \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k}) \cdot dt$$

$$= \int_0^1 3t^2 + 4t^5 - 2t^3 + 3t^5 \cdot dt$$

$$= \left[\frac{7t^6}{6}\right]_0^1 - 2\left[\frac{t^4}{4}\right]_0^1 + 3\left[\frac{t^3}{3}\right]_0^1$$

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$$\oint \bar{F}\cdot\frac{d\bar{r}}{dt} = \frac{5}{3}$$

3 Find the circulation around the circle $\bar{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$ of the velocity field $\bar{F} = (x-y)\,\hat{i} + x\,\hat{j}$, $0 \leq t \leq 2\pi$.

Let, position vectors,

$$\bar{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$$

$$\therefore \frac{d\bar{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$\Rightarrow \bar{F} = (x-y)\,\hat{i} + x\,\hat{j}$$

$$= (a\cos t - a\sin t)\,\hat{i} + (a\cos t)\,\hat{j}$$

$$\Rightarrow \oint \bar{F}\cdot\frac{d\bar{r}}{dt} = \int_0^{2\pi} (a\cos t - a\sin t)\,\hat{i} + a\cos t\,\hat{j} \cdot (-a\sin t\,\hat{i} + a\cos t\,\hat{j})\cdot dt$$

$$= \int_0^{2\pi} a^2(\sin^2 t + \cos^2 t) - a^2\sin t\cdot\cos t \cdot dt$$

$$= \int_0^{2\pi} a^2 - \frac{a^2}{2} \sin 2t \cdot dt$$

$$= a^2 \left[t - \frac{\cos 2t}{4} \right]_0^{2\pi}$$

$$= 2\pi a^2$$

4 If $\bar{F} = 2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$ then,

(a) If $\bar{F}$ is conservative. Find its scalar potential ϕ

$\Rightarrow$ Here, $\bar{F} = \bar{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

where, $\frac{\partial \phi}{\partial x} = 2xyz$, $\frac{\partial \phi}{\partial y} = x^2z + 2y$

$$\frac{\partial \phi}{\partial z} = x^2 y$$

$\rightarrow$ $\phi_{(x)} = x^2yz + f(x,y)$

$\phi_{(y)} = x^2z + y^2 + f(x,y)$

$\phi_{(z)} = x^2yz + f(x,y)$

$\therefore \phi = x^2yz + y^2 + C$

(b) Find the work done in moving a particle under this force field from (0, 1, 1) to (1, 2, 0)

$\therefore$ Work done $= \int_C d\phi$

$\therefore W = \int_{(0,1,1)}^{(1,2,0)} d\phi$

$\therefore W = \left[x^2yz + y^2\right]_{(0,1,1)}^{(1,2,0)}$

$= 4 - 1$

$\therefore W = 3$

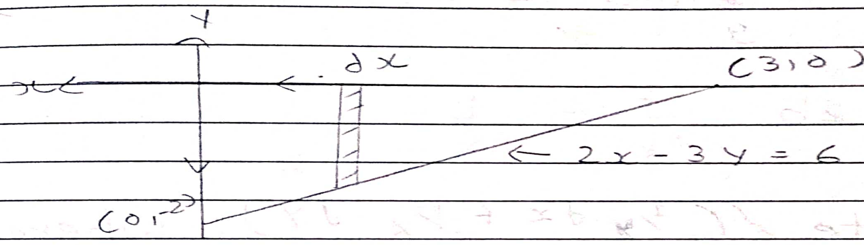
5 Using Green's theorem evaluate $\int_C (3x^2 - 8y^2)\cdot dx + (4y - 6xy)\cdot dy$ where c is the bounded by $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$

$\rightarrow$ $M_{(x,y)} = 3x^2 - 8y^2$

$N_{(x,y)} = 4y - 6xy$

$\therefore \dfrac{\partial M}{\partial y} = -16y$ and $\dfrac{\partial N}{\partial x} = -6y$

$\therefore I = \iint_R (-6y + 16y)\, dx\, dy$



$\therefore I = \iint_R 10y \cdot dy\, dx$

$\therefore I = 10 \int_0^3 \int_0^{2x-6/3} y \, dy\, dx$

$= 10 \int_0^3 \left[\dfrac{y^2}{2} \right]_0^{2x-6/3} dx$

$\therefore \dfrac{10}{2} \int_0^3 \dfrac{(2x-6)^2}{9}\, dx$

$$= \frac{5}{9}\int_0^3 4x^2 - 24x + 36 \cdot dx$$

$$= \frac{5}{9}\left[\frac{4}{3}x^3 - \frac{24x^2}{2} + 36x\right]_0^3$$

$$= \frac{5}{9}\left[9 \times 4 \times 1\right]$$

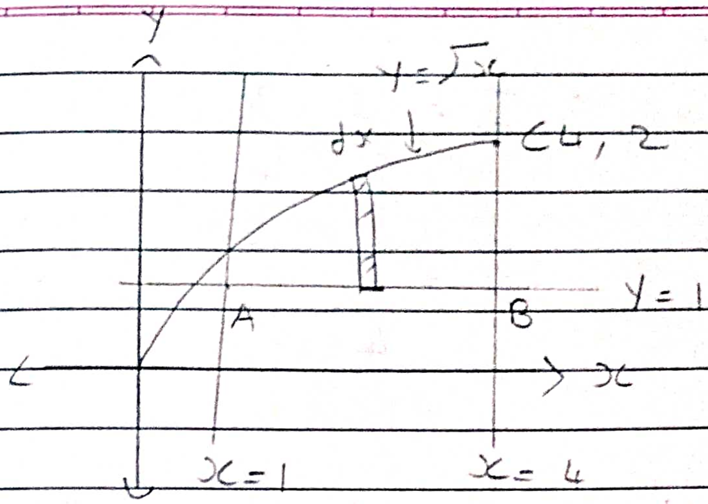
$$I = 20$$

6 Evaluate $\oint_C \left(\frac{1}{y}\,dx + \frac{1}{x}\cdot dy\right)$; where c is the boundary of the region bounded by the parabola $y = \sqrt{x}$ and the lines $x = 1$, $x = 4$, $y = 1$ and also verify the Green's theorem.

$$I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \quad — (1)$$

$\rightarrow$ $M_{(x,y)} = \frac{1}{y} \Rightarrow \frac{\partial M}{\partial y} = \frac{-1}{y^2}$

$N_{(x,y)} = \frac{1}{x} \Rightarrow \frac{\partial N}{\partial x} = \frac{-1}{x^2}$



$$\therefore I = \iint_R -\frac{1}{x^2} + \frac{1}{y^2} \cdot dx\,dy$$

$$\therefore I = \int_1^4 \int_{y=1}^{\sqrt{x}} -\frac{1}{x^2} + \frac{1}{y^2}\, dy \cdot dx$$

$$\therefore I = \int_1^4 \left[-\frac{y}{x^2} - \frac{1}{y} \right]_1^{\sqrt{x}} dx$$

$$\therefore I = \int_1^4 -\frac{\sqrt{x}}{x^2} - \frac{1}{\sqrt{x}} - \left[\frac{-1}{x^2} - 1 \right] dx$$

$$I = \int_1^4 -x^{-3/2} - x^{-1/2} + x^{-2} + 1 \cdot dx$$

$$\therefore I = \left[2x^{-1/2} - 2x^{1/2} - x^{-1} + x \right]_1^4$$

$$\therefore I = 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

7 Find the work done by force $\bar{F} = (4x-2y)\hat{i} + (2x-4y)\hat{j}$ in moving a particle once counter clockwise around the circle $(x-2)^2 + (y-2)^2 = 4$ Using green's theorem.

=>

$$I = \oint_C M(x,y) + N(x,y)\, dx\, dy$$

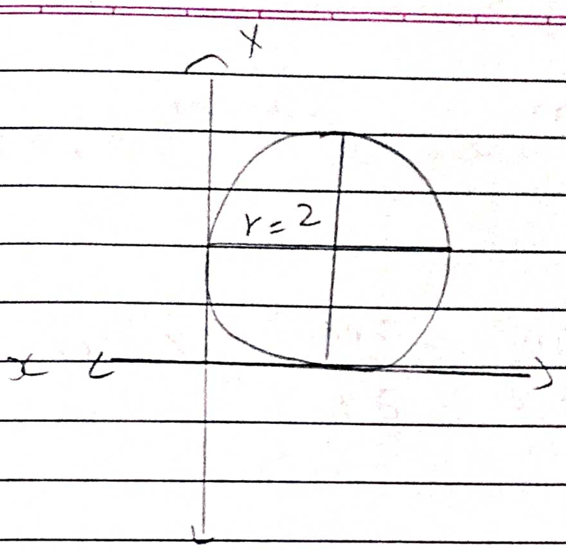
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

-> $M = 4x - 2y$

$$\frac{\partial m}{\partial y} = -2$$

-> $N = 2x - 4y$

$$\frac{\partial m}{\partial x} = 2$$



$$I = \iint_R 2-(-2)\,dx\,dy$$

$$= \iint_R 4\,dx\,dy$$

$$= 4\iint_R \pi r^2 \cdot dx\,dy$$

$$= 4\pi(2)^2$$

$$I = 16\pi$$

8 Using green's theorem to evaluate $\oint_c (2xy)\,dx - (y^2)\,dy$, where c is the boundary of the region bounded by ellipe $\frac{x^2}{a^2}+\frac{y^2}{6^2}=1$

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$\Rightarrow I = \oint_C M_{(x,y)} + N_{(x,y)} \cdot dx\,dy$

$I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$ — ①

$\rightarrow M_{(x,y)} = 2xy$

$\frac{\partial M}{\partial y} = 2x$

$\rightarrow N_{(x,y)} = -y^2$

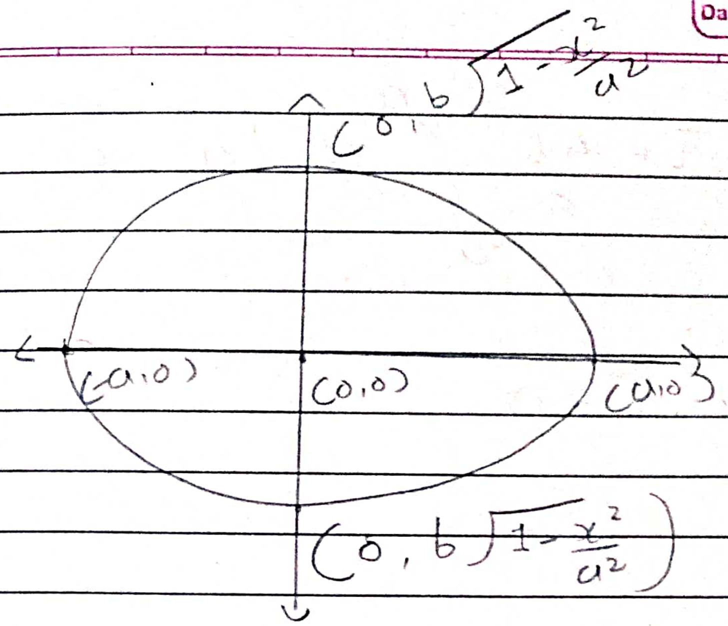
$\frac{\partial N}{\partial x} = 0$

put all value in $eq^n - 1$

$\therefore I = \iint_R (0 - 2x)\,dx\,dy$

limit : x : $-a$ to a

limit : y : $-b\sqrt{1 - \frac{x^2}{a^2}}$ to $b\sqrt{1 - \frac{x^2}{a^2}}$



$$\therefore\ I = -\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} 2x\, dy\, dx$$

$$\therefore\ I = -\int_{-a}^{a} \left[2xy\right]_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \cdot dx$$

$$\therefore\ I = -\int_{-a}^{a} 2x\left[b\sqrt{1-\frac{x^2}{a^2}} + b\sqrt{1-\frac{x^2}{a^2}}\right]\cdot dx$$

$$\therefore\ I = -\int_{-a}^{a} \frac{2x \cdot 2b\sqrt{a^2-x^2}}{a^2}\cdot dx$$

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$$I = \frac{4b}{a}\int_{-a}^{a} x\sqrt{a^2-x^2}\cdot dx$$

$\therefore\ I = 0$