Unit-2 Partial Differentiation and its Applications

* Task 1: Limit of function of two variables.

1 $\lim_{(x,y)\to(0,0)} \frac{4y^2x}{x^2+y^2}$

$\to \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{4y^2x}{x^2+y^2}\right\}$

$= \lim_{x\to 0} 0$

$= 0 \quad — \quad (1)$

$\to \lim_{y\to 0}\left\{\lim_{x\to 0} \frac{4y^2x}{x^2+y^2}\right\}$

$= \lim_{y\to 0} 0$

$= 0 \quad — \quad (2)$

$\to$ putting $y = mx$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{4x(mx)^2}{x^2+(mx)^2}$$

$$= \lim_{x \to 0} \frac{4mx^3}{x^2(1+m^2)}$$

$$= \lim_{x \to 0} \frac{4mx}{1+m^2}$$

$$= 0 \quad - \quad (3)$$

→ putting $y = mx^2$, taking $x \to 0$

$$= \lim_{x \to 0} \frac{4x(mx^2)^2}{x^2+(mx^2)^2}$$

$$= \lim_{x \to 0} \frac{4x^5 m}{x^2+mx^4}$$

$$= \lim_{x \to 0} \frac{4x^3 m}{1+mx^2}$$

$$= 0 \quad - \quad (4)$$

From eq^n 1, 2, 3 and 4 values are equal

So $\lim_{(x,y)\to(0,0)} \frac{4y^2x}{x^2+y^2}$ exist.

2 $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

$\to \lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \right\}$

$= \lim_{x\to 0} \frac{x^2}{\sqrt{x}}$

$= 0 \quad - \quad 1$

$\to \lim_{x\to 0} \left\{ \lim_{x\to 0} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \right\}$

$= \lim_{y\to 0} x \frac{1}{\sqrt{y}}$

$= 0 \quad - \quad 2$

$\to$ putting $y = mx$, taking $x \to 0$

$= \lim_{x\to 0} \frac{x^2 - x(mx)}{\sqrt{x} - \sqrt{mx}}$

$= 0 \quad - \quad 3$

$\to$ putting $y = mx^2$ taking $x \to 0$

$= \lim_{x\to 0} \frac{x^2 - x(mx^2)}{\sqrt{x} - \sqrt{mx^2}}$

$= 0$ — (4)

From, eq^n 1, 2, 3 and 4 value are equal so, $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ exist

3 $\lim_{(x,y)\to(2,1)} \frac{x^2+2y}{x+y^2}$

$\to \lim_{x\to 2} \left\{ \lim_{y\to 1} \frac{x^2+2y}{x+y^2} \right\}$

$= \lim_{x\to 2} \frac{x^2+2}{x+1}$

$= \frac{4+2}{2+1}$

$= \frac{6}{3}$

$= 2$ — (1)

$\to \lim_{y\to 1} \left\{ \lim_{x\to 2} \frac{x^2+2y}{x+y^2} \right\}$

$$= \lim_{y \to 1} \frac{4+2y}{2+y^2}$$

$$= \frac{4+2}{2+1}$$

$$= \frac{6}{3}$$

$$= 2 \quad — \quad (2)$$

From eq^n 1 and 2 value are equal

So, $\lim_{(x,y) \to (1,2)} \frac{x^2+2y}{x+y^2}$ exist at 2

4 $\lim_{(x,y) \to (0,0)} \frac{x^3-y^3}{x^2+y^2}$

$\to \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x^3-y^3}{x^2+y^2} \right\}$

$$= \lim_{x \to 0} \frac{x^3}{x^2}$$

$$= 0 \quad — \quad (1)$$

$\to \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x^3-y^3}{x^2+y^2} \right\}$

$$= \lim_{y \to 0} \frac{-y^3}{y^2}$$

$$= 0 \quad — \text{(2)}$$

→ putting $y = mx$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{x^3 - m^3x^3}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{x^2(x - m^3x)}{x^2(1 + m^2)}$$

$$= \lim_{x \to 0} \frac{x - m^3x}{1 + m^2}$$

$$= 0 \quad — \text{(3)}$$

→ putting $y = mx^2$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{x^3 - m^3x^4}{x^2 + m^2x^3}$$

$$= \lim_{x \to 0} \frac{x^3(x^2 - m^3x)}{x^2(1 + mx)}$$

$$= \lim_{x \to 0} \frac{x^2 - m^3x}{1 + mx}$$

$$= 0 \quad — \text{(4)}$$

From eq^n 1, 2, 3 and 4 value are equal.

So, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$ limit exist

5 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

$\rightarrow \lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \dfrac{x^2y}{x^4+y^2} \right\}$

$= 0 \quad - \quad (1)$

$\rightarrow \lim\limits_{y\to 0} \left\{ \lim\limits_{x\to 0} \dfrac{x^2y}{x^4+y^2} \right\}$

$= 0 \quad - \quad (2)$

$\rightarrow$ putting $y = mx$ taking $x \to 0$

$= \lim\limits_{x\to 0} \dfrac{x^2(mx)}{x^4+m^2x^2}$

$= \lim\limits_{x\to 0} \dfrac{x^3 m}{x^2(x^2+m^2)}$

$= \lim\limits_{x\to 0} \dfrac{xm}{1+m^2}$

$= 0$ — (3)

→ putting $y = mx^2$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{x^2(mx^2)}{x^4 + m^2x^4}$$

$$= \lim_{x \to 0} \frac{x^4 m}{x^3(x + m^2)}$$

$$= \lim_{x \to 0} \frac{xm}{x + m^2}$$

$= 0$ — (4)

Form eq^n 1, 2, 3 and 4 value are equal.
So, $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$ exist at 0.

6 $$f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & : \text{if } (x,y) \neq (0,0) \\ 0 & : \text{if } (x,y) = (0,0) \end{cases}$$

given $f(0,0) = 0$

→ $$\lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x}{\sqrt{x^2+y^2}} \right\}$$

$= \lim_{x \to 0} \frac{x}{\sqrt{x}}$

$= 0 \quad — \quad (1)$

$\Rightarrow \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x}{\sqrt{x^2+y^2}} \right\}$

$= 0 \quad — \quad (2)$

$\Rightarrow$ putting $y = mx$ taking $x \to 0$

$= \lim_{x \to 0} \frac{x}{\sqrt{x^2+m^2x^2}}$

$= \lim_{x \to 0} \frac{x}{x\sqrt{1+m^2}}$

$= \frac{1}{\sqrt{1+m^2}}$

Here, last limit depends on m and m is not Fix.

So, $\lim_{(x,y) \to (0,0)}$ is not exist at 0.

* Task : 2 Continuity of function of two variables.

1 $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & ; \text{ if } (x,y) \neq (0,0) \\ 0 & ; \text{ if } (x,y) = (0,0) \end{cases}$

$\rightarrow \lim\limits_{(x,y)\to(0,0)} = \dfrac{x^3-y^3}{x^2+y^2}$

$\rightarrow \lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \dfrac{x^3-y^3}{x^2+y^2} \right\}$

$= \lim\limits_{x\to 0} \dfrac{x^3}{x^2}$

$= \lim\limits_{x\to 0} x$

$= 0 \quad - (1)$

$\rightarrow \lim\limits_{y\to 0} \left\{ \lim\limits_{x\to 0} \dfrac{x^3-y^3}{x^2+y^2} \right\}$

$= \lim\limits_{y\to 0} \dfrac{-y^3}{y^2}$

$= \lim\limits_{y\to 0} -y$

In this
to initilize

$= 0 \quad - \quad (2)$

→ putting $y = mx$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{x^3 - m^3x^3}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{x^2(x - m^3x)}{x^2(1 + m^2)}$$

$$= \lim_{x \to 0} \frac{x - m^3x}{1 + m^3}$$

$= 0 \quad - \quad (3)$

→ putting $y = mx^2$ taking $x \to 0$

$$= \lim_{x \to 0} \frac{x^3 - m^3x^4}{x^2 + m^2x^{\cancel{4}3}}$$

$$= \lim_{x \to 0} \frac{x^2(x - x^2m^3)}{x^2(1 + mx)}$$

$$= \lim_{x \to 0} \frac{x - x^2m^3}{1 + mx}$$

$= 0 \quad - \quad (4)$

From eq^n 1, 2, 3 and 4 are equal

So $\lim_{(x,y) \to (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$ exist.

hanse $F(x, y)$ Continues at $x=0$.

2 $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & ; \text{ if } (x,y) \neq (0,0) \\ 4 & ; \text{ if } (x,y) = (0,0) \end{cases}$

given $f(0,0) = 4$

$\rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

$\rightarrow \lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \dfrac{x^2-y^2}{x^2+y^2} \right\}$

$= \lim\limits_{x\to 0} \dfrac{x^2}{x^2}$

$= 1$ — (1)

$\rightarrow \lim\limits_{y\to 0} \left\{ \lim\limits_{x\to 0} \dfrac{x^2-y^2}{x^2+y^2} \right\}$

$= \lim\limits_{y\to 0} \dfrac{-y^2}{y^2}$

$= -1$ — (2)

$F(x,y)$ is discontinues at $(0,0)$.

3 $F(x, y) = \begin{cases} \dfrac{x^2+2y}{x+y^2} & ; \text{ if } (x, y) \neq (2, 1) \\ 2 & ; \text{ if } (x, y) = (2, 1) \end{cases}$

given, $F(2, 1) = 2$

$\rightarrow \lim_{(x,y)\rightarrow(2,1)} \dfrac{x^2+2y}{x+y^2}$

$\rightarrow \lim_{x\rightarrow 2} \left\{ \lim_{y\rightarrow 1} \dfrac{x^2+2y}{x+y^2} \right\}$

$= \lim_{x\rightarrow 2} \dfrac{x^2+2}{x+1}$

$= \dfrac{6}{3}$ — 1

$\rightarrow \lim_{y\rightarrow 1} \left\{ \lim_{x\rightarrow 2} \dfrac{x^2+2y}{x+y^2} \right\}$

$= \lim_{y\rightarrow 1} \dfrac{4+2(y)}{2+y^2}$

$= \dfrac{6}{3}$ — 2

$F(x, y)$ is continues at $(2, 1)$

$\lim_{(x,y)\rightarrow(2,1)} F(x, y) = F(2, 1)$

$F(x$

4 $F(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 0 & ; \text{if } (x,y) = (0,0) \end{cases}$

given $F(0,0) = 0$

$\rightarrow \lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x^2+y^2}$

$\rightarrow \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x^2-y^2}{x^2+y^2} \right\}$

$= \lim_{x \to 0} \frac{x^2}{x^2}$

$= 1$

$\rightarrow \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x^2-y^2}{x^2+y^2} \right\}$

$= \lim_{y \to 0} \frac{-y^2}{y^2}$

$= -1$

$F(x,y)$ is discontinues at $(0,0)$

$\lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x^2+y^2} \neq F(0,0)$

* Task 6: Partial Differentiation of Composite Functions.

1 If $\tan^{-1}\left(\frac{x}{y}\right)$ and $x = 2t$ and $y = 1 - t^2$

then prove that $\frac{\partial z}{dt} = \frac{2}{1+t^2}$

$$
\begin{tikzcd}
 & x \arrow[r, "\frac{dx}{dt}"] & t \\
z \arrow[ur, "\frac{\partial z}{\partial x}"] \arrow[dr, "\frac{\partial z}{\partial y}"'] & & \\
 & y \arrow[r, "\frac{dy}{dt}"] & t
\end{tikzcd}
$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} \cdot 2 + \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right)(-2t)$$

$$= \frac{2y^2}{x^2+y^2} \cdot \frac{1}{y} + \frac{y^2}{x^2+y^2} \cdot \frac{2xt}{y^2}$$

$$= \frac{2y}{x^2+y^2} + \frac{2xt}{x^2+y^2}$$

$$= \frac{2(y+xt)}{x^2+y^2}$$

$$\frac{dz}{dt} = \frac{2\left(1 - t^2 + 2t(t)\right)}{4t^2 + 1 - 2t^2 + t^4}$$

$$= \frac{2\left(1 - t^2 + 2t^2\right)}{2t^2 + 1 + t^4}$$

$$= \frac{2\left(1 + t^2\right)}{\left(1 + t^2\right)^2}$$

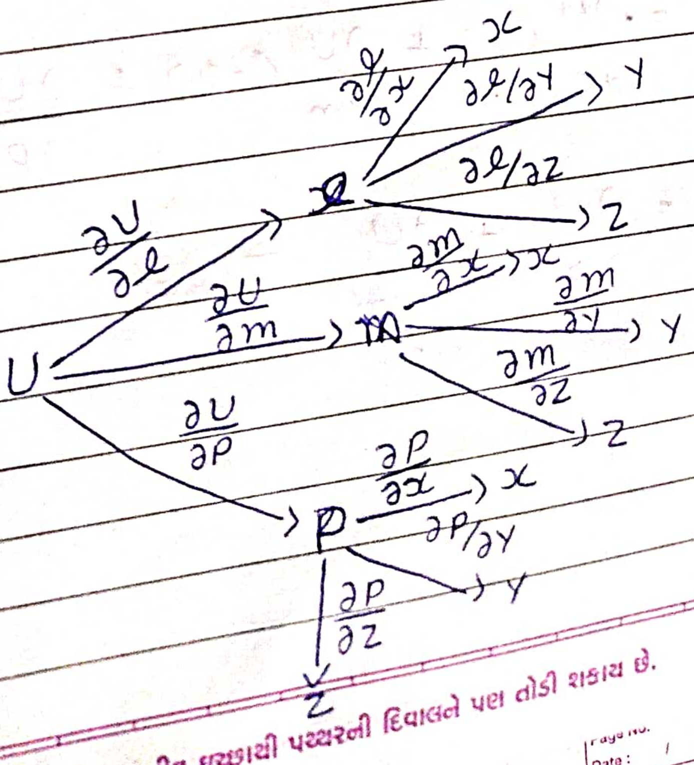
$$= \frac{2}{1 + t^2}$$

2 If $U = F(x - y,\ y - z,\ z - x)$ then prove tha

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$

$x - y = l$, $y - z = m$

$z - x = p$



$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial x}$$

$$= \frac{\partial u}{\partial l}(1) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial p}(-1)$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial p} \quad — (1)$$

$$\rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-1) + \frac{\partial u}{\partial m}(1) + \frac{\partial u}{\partial p}(0)$$

$$= \frac{\partial u}{\partial m} - \frac{\partial u}{\partial l} \quad — (2)$$

$$\rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial z}$$

$$= \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial p}(1)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial m} \quad — (3)$$

eq^n 1 + 2 + 3,

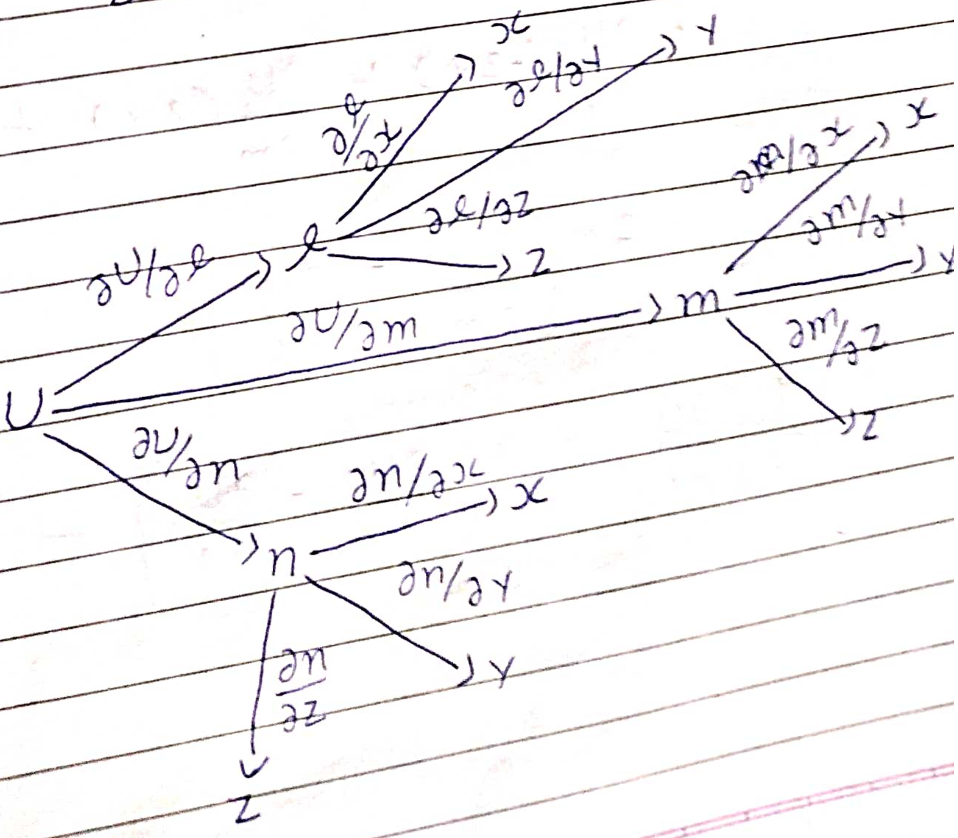
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial m} - \frac{\partial u}{\partial l} + \frac{\partial u}{\partial p} - \frac{\partial u}{\partial m}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

3) If $U = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

We take, $x^2 - y^2 = l$
$y^2 - z^2 = m$
$z^2 - x^2 = n$



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$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial x}$

$= \frac{\partial u}{\partial l} \times 2x + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-2x)$

$= \frac{\partial u}{\partial l} 2x - 2x \cdot \frac{\partial u}{\partial n}$

$= 2x\left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right)$

$\frac{1}{x}\left(\frac{\partial u}{\partial x}\right) = 2\left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right) \quad — (1)$

$\rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial y}$

$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$

$= 2y\left(\frac{\partial u}{\partial m} - \frac{\partial u}{\partial l}\right)$

$\frac{1}{y}\left(\frac{\partial u}{\partial y}\right) = 2\left(\frac{\partial u}{\partial m} - \frac{\partial u}{\partial l}\right) \quad — (2)$

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$$\rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

$$= 2z\left(\frac{\partial u}{\partial n} - \frac{\partial u}{\partial m}\right)$$

$$\frac{1}{z}\left(\frac{\partial u}{\partial z}\right) = 2\left(\frac{\partial u}{\partial n} - \frac{\partial u}{\partial m}\right) \quad \text{--- (3)}$$

eq^n 1 + 2 + 3,

$$= \frac{1}{x}\left(\frac{\partial u}{\partial x}\right) + \frac{1}{y}\left(\frac{\partial u}{\partial y}\right) + \frac{1}{z}\left(\frac{\partial u}{\partial z}\right)$$

$$= 2\left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right) + 2\left(\frac{\partial u}{\partial m} - \frac{\partial u}{\partial l}\right) + 2\left(\frac{\partial u}{\partial n} - \frac{\partial u}{\partial m}\right)$$

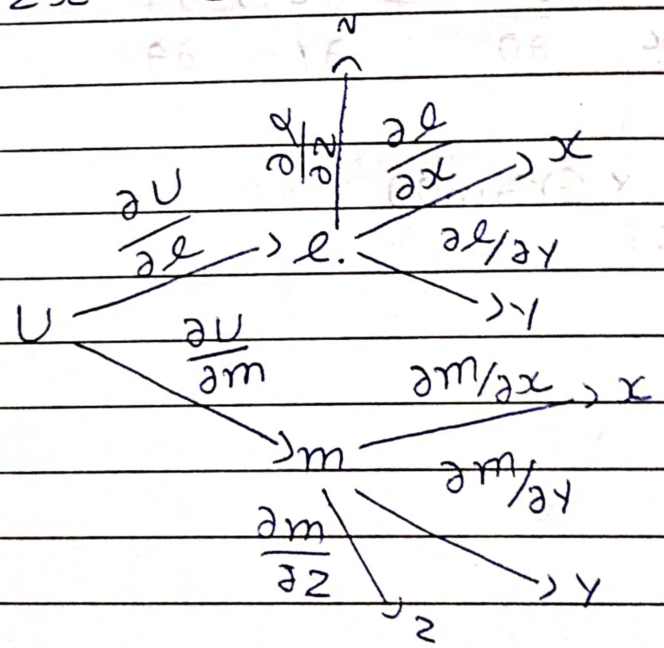
$$= 0$$

5 If $U = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ then prove that, $x^2 \frac{\partial U}{\partial x} + y^2 . \frac{\partial U}{\partial y} + z^2 . \frac{\partial U}{\partial z} = 0$

Here we take $\frac{y-x}{xy} = l$, $\frac{z-x}{zx} = m$

$\Rightarrow l = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$

$\Rightarrow m = \frac{z}{zx} - \frac{x}{zx} = \frac{1}{x} - \frac{1}{z}$



$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x}$$

$$= \frac{\partial U}{\partial l}\left(\frac{-1}{x^2}\right) + \frac{\partial U}{\partial m}\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial l} - \frac{\partial u}{\partial m} \quad \text{—} \quad (1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left[+\frac{1}{y^2}\right] + \frac{\partial u}{\partial m}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} \cdot y^2 = +\frac{\partial u}{\partial l} \quad \text{—} \quad (2)$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}\left[+\frac{1}{z^2}\right]$$

$$\Rightarrow z^2 \cdot \frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \quad \text{—} \quad (3)$$

From eq^n 1 + 2 + 3,

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial l} - \frac{\partial u}{\partial m} + \frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}$$

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z} = 0$$

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* Task 3: Partial Differentiation of function of two or more variables.

1 By using the defination find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, $f(x, y, z) = x^2yz^2$ at point $(1, 2, 3)$

$\rightarrow \frac{\partial f}{\partial x} = 2x \cdot y \cdot z^2$

$\left(\frac{\partial f}{\partial x}\right)_{(1,2,3)} = 2(1) \cdot 2 \cdot (3)^2 = 36$

$\rightarrow \frac{\partial f}{\partial y} = x^2 z^2$

$\left(\frac{\partial f}{\partial y}\right)_{(1,2,3)} = (1)^2 \cdot (3)^2 = 9$

$\rightarrow \frac{\partial f}{\partial z} = x^2 y \cdot 2z$

$\left(\frac{\partial f}{\partial z}\right)_{(1,2,3)} = (1)^2 + 2 \cdot 2(3) = 12$

2. If $U = e^{xyz}$ then Find $\dfrac{\partial^3 U}{\partial x \partial y \partial z}$

$\rightarrow \dfrac{\partial U}{\partial z} = e^{xyz} \cdot xy$

$\rightarrow \dfrac{\partial}{\partial y}\left(\dfrac{\partial U}{\partial z}\right) = \dfrac{\partial^2 U}{\partial y \partial z} = e^{xyz} \cdot xz \cdot xy + e^{xyz} \cdot x$

$\rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 U}{\partial y \partial z}\right) = e^{xyz} \cdot yz \cdot xz \cdot xy + e^{xyz} \cdot z \cdot xy$

$+ e^{xyz} \cdot xz \cdot y + e^{xyz} \cdot yz \cdot x$

$+ e^{xyz}$

$= e^{xyz} \cdot x^2 y^2 z^2 + e^{xyz} \cdot 3xyz + e^{xyz}$

$\dfrac{\partial^3 U}{\partial x \partial y \partial z} = e^{xyz}(1 + 3xyz + x^2 y^2 z^2)$

3. If $U = \tan(y + ay) + (y - ax)^{3/2}$ then show that, $\dfrac{\partial^2 U}{\partial x^2} = a^2 \dfrac{\partial^2 U}{\partial y^2}$

Given $U = \tan(y + ay) + (y - ax)^{3/2}$

$\rightarrow \dfrac{\partial U}{\partial x} = \sec^2(y + ax)a + \dfrac{3}{2}(y - ax)^{1/2}(-a)$

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} = 2a\sec(Y+aY)\cdot\sec(Y+a
\cdot\tan(Y+aY) + \frac{3}{4}(Y-ax)^{-\frac{1}{2}}\cdot a^2$$

$$\rightarrow \frac{\partial^2 U}{\partial x^2} = 2a\sec^2(Y+ax)\cdot\tan(Y+ax) + \frac{3}{4}(Y-ax)^{-\frac{1}{2}}\cdot a^2 \quad — (1)$$

$$\rightarrow \frac{\partial U}{\partial Y} = \sec^2(Y+ax)\cdot a + \frac{3}{2}(Y-ax)^{\frac{1}{2}}$$

$$\rightarrow \frac{\partial^2 U}{\partial y^2} = 2\sec^2(Y+ax)\cdot\tan(Y+ax) + \frac{3}{4}(Y-ax)^{-1/2}\cdot a^2 \quad — (2)$$

From eq^n 1 and 2

$\therefore \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial y^2}$ is equal.

4 If $U = e^{x^2+y^2+z^2}$ then prove that

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = 8xyzU$$

$$\rightarrow \frac{\partial U}{\partial z} = e^{x^2+y^2+z^2}(2z)$$

$$\rightarrow \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z}\right) = \frac{\partial^2 U}{\partial y \partial z} = e^{x^2+y^2+z^2}(2z)(2y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 U}{\partial y \partial z}\right) = \frac{\partial^3 U}{\partial x \partial y \partial z} = e^{x^2+y^2+z^2}(2z)(2y)(2x)$$

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = 8xyzU$$

5 If $U = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then prove that $U_{xy} = U_{yx}$

$$\text{L.H.S.} = U_{xy} = \frac{\partial^2 U}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} + 0 \times \tan^{-1}\left(\frac{y}{x}\right)$$

$$-2y \cdot \tan^{-1}\left(\frac{x}{y}\right) + y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \left(\frac{-x}{y^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\rightarrow \frac{\partial u}{\partial y} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\rightarrow \frac{\partial^2 u}{\partial x \partial y} = 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \quad \text{--- } (1)$$

$$\text{R.H.S.} = u_{yx} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\rightarrow \frac{\partial u}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right)$$

$$- y^2 \cdot \frac{1}{1+x^2/y^2} \cdot 1/y$$

$$\frac{\partial u}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - y$$

$$\rightarrow \frac{\partial^2 u}{\partial y \partial x} = 2x \cdot \frac{1}{1+y^2/x^2} \cdot 1/x - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1$$

$$= \frac{x^2-y^2}{x^2+y^2} \quad \text{—} \quad (2)$$

$\rightarrow$ By eq^n 1 and 2 L.H.S. = R.H.S.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

* Task 4 : Partial Differentiation of function of function.

1 If $U = \log(x^3+y^3+z^3-3xyz)$ then prove that,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)U$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}\right)$$

Here, we take $\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = V$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot V \quad - (1)$$

$$\Rightarrow V = \frac{(3x^2-3yz)}{x^3+y^3+z^3-3xyz} + \frac{(3y^2-3xz)}{x^3+y^3+z^3-3xyz}$$

$$+ \frac{(3z^2-3xy)}{x^3+y^3+z^3-3xyz}.$$

$$V = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore v = \frac{3(x^3+y^3+z^3-yz-xy-xz)}{x^3+y^3+z^3-3xyz} \times \frac{(x+y+z)}{(x+y+z)}$$

$$\therefore v = 3\,\frac{(x^3+y^3+z^3-3xyz)}{x^3+y^3+z^3-3xyz(x+y+z)}$$

$$\therefore v = \frac{3}{x+y+z}$$

put v value in eq^n 1

$$\therefore \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{x+y+z}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u \text{ is prove.}$$

2 If $u = \log(x^3+y^3-x^2y-xy^2)$ then prove that,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

Here we take $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = v$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) v \quad \text{— (1)}$$

$$\rightarrow \quad v = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$$

$$= \frac{(3x^2-2xy-y^2)}{x^3+y^3-x^2y-xy^2} + \frac{(3y^2-x^2-2xy)}{x^3+y^3-x^2y-xy^2}$$

$$= \frac{2x^2+2y^2-2xy-2xy}{x^3+y^3-x^2y-xy^2}$$

$$v = \frac{-2x(-x+y)+2y(y-x)}{-x^2(y-x)+y^2(y-x)}$$

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$$\therefore V = \frac{(2y-2x)(y-x)}{y^2-x^2(y-x)}$$

$$\therefore V = \frac{2(y-x)}{(y-x)(y+x)}$$

$$\therefore V = \frac{2}{y+x}$$

~~putting v value in eq^n 1~~

$\therefore$ both side log,

$$\therefore \log V = x \log 2 - \log(y+x)$$

derivative w.r.t to x

$$\therefore \frac{1}{v}\left(\frac{\partial v}{\partial x}\right) = -\frac{1}{y+x} \quad - (2)$$

derivative w.r.t to y

$$\therefore \frac{1}{v}\left(\frac{\partial v}{\partial y}\right) = -\frac{1}{y+x} \quad - (3)$$

adding eq^n 2 and 3

$$\therefore \frac{1}{v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{-2}{y+x}$$

$$\therefore \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = \frac{-2v}{(y+x)}$$

$$\therefore \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = \frac{-4}{(y+x)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$$ is prove.

3 If $u = f(r)$, where $r^2 = x^2 + y^2$ then prove,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

=) $r^2 = x^2 + y^2$

$u = f(r)$ is given,

$$\therefore \frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

derivative w.r. to x

$$\therefore \frac{\partial^2 u}{\partial x^2} = f''(r) \frac{\partial r}{\partial x} + f'(r) \cdot \frac{1}{r} - f'(r) \frac{x}{r^2} \left(\frac{\partial r}{\partial x}\right)$$

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$$\therefore \frac{\partial^2 U}{\partial x^2} = f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{x^2}{r^3} \quad — (1)$$

Similarly,

$$\therefore \frac{\partial^2 U}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{y^2}{r^3} \quad — (2)$$

adding eq^n 1 and 2,

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f''(r)\frac{(x^2+y^2)}{r^2} + 2\frac{f'(r)}{r}$$

$$- f'(r)\frac{(x^2+y^2)}{r^3}$$

$$= f''(r)\frac{(r^2)}{r^2} + \frac{2}{r}f'(r)$$

$$- f'(r)\frac{(r^2)}{r^3}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

4 If $U = f(r^2)$ where $r^2 = x^2+y^2+z^2$ then prove,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$$

→ Here, $U = f(r^2)$

$$\therefore \frac{\partial U}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{\partial r}{\partial x}$$

→ $r^2 = x^2 + y^2 + z^2$

$$\therefore 2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

→ $$\frac{\partial U}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{x}{r}$$

$$\frac{\partial U}{\partial x} = f'(r^2) \cdot 2x$$

~~da~~ derivitive w.r.t to x

$$\therefore \frac{\partial^2 U}{\partial x^2} = f''(r^2) \cdot 2r \cdot \frac{\partial r}{\partial x} \cdot 2x + 2f'(r^2)$$

$\therefore \frac{\partial^2 U}{\partial x^2} = 4x^2 f''(r^2) + 2f'(r^2)$ — 1

Similarly,

$\frac{\partial^2 U}{\partial y^2} = 4y^2(f''(r^2)) + 2f'(r^2)$ — 2

$\frac{\partial^2 U}{\partial z^2} = 4z^2(f''(r^2)) + 2f'(r^2)$ — 3

adding eq^n 1, 2 and 3

$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 4f''(r^2)(x^2+y^2+z^2) + 6f'(r^2)$

$= 4f''(r^2)r^2 + 6f'(r^2)$

5 If $U = \log(\tan x + \tan y + \tan z)$ then prove that,

$\sin(2x)U_x + \sin(2y)U_y + \sin(2z)U_z = 2$

=) $U = \log(\tan x + \tan y + \tan z)$

$\frac{\partial U}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$

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$\rightarrow$ $\sin(2x) \cdot \frac{\partial U}{\partial x} = \frac{2\sin x \cdot \cos x}{\cos^2 x(\tan x + \tan y + \tan z)}$

$\sin(2x) \frac{\partial U}{\partial x} = \frac{2\tan x}{\tan x + \tan y + \tan z}$ — (1)

Similarly,

$\rightarrow$ $\sin(2y) \frac{\partial U}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$ — (2)

$\rightarrow$ $\sin(2z) \cdot \frac{\partial U}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$ — (3)

adding eq^n 1, 2 and 3

$\therefore$ $\sin 2x \cdot \frac{\partial U}{\partial x} + \sin 2y \cdot \frac{\partial U}{\partial y} + \sin 2z \left(\frac{\partial U}{\partial z}\right)$

$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$

$= 2$

6 If $U = (x^2+y^2+z^2)^{1/2}$ then prove that,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = -U$$

$\rightarrow$ $$U = (x^2+y^2+z^2)^{1/2}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{-1}{2} \frac{(2x)}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$\rightarrow$ $$x \cdot \frac{\partial U}{\partial x} = \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{— (1)}$$

Similarly,

$\rightarrow$ $$y \cdot \frac{\partial U}{\partial \cancel{x}} = \frac{-y^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{— (2)}$$

$\rightarrow$ $$z \cdot \frac{\partial U}{\partial z} = \frac{-z^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{— (3)}$$

adding eq^n 1, 2 and 3,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = \frac{-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = - \frac{1}{(x^2+y^2+z^2)^{-1/2}}$$

$$\therefore x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = -U$$ is prove.

* Task : 5 Partial Differentiation of Function in which some variable treated as constant.

1 If $x = r\cos\theta$ and $y = r\sin\theta$ then Find,

(a) $\left(\frac{\partial x}{\partial r}\right)_\theta$ (b) $\left(\frac{\partial y}{\partial \theta}\right)_r$ (c) $\left(\frac{\partial r}{\partial x}\right)_y$ (d) $\left(\frac{\partial \theta}{\partial y}\right)_x$

=> $x^2 = r^2\cos^2\theta$

$y^2 = r^2\sin^2\theta$

$\therefore\ x^2 + y^2 = r^2\sin^2\theta + r^2\cos^2\theta$

$\therefore\ x^2 + y^2 = r^2$

a $\left(\frac{\partial x}{\partial r}\right)_\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

b $\left(\frac{\partial y}{\partial \theta}\right)_r$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

c $\left(\frac{\partial r}{\partial x}\right)_y$

$$x^2 + y^2 = r^2$$

$$\therefore 2x = 2r \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

d $\left(\frac{\partial \theta}{\partial y}\right)_x$

$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\therefore \tan\theta = \frac{y}{x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2 + y^2}$$

2 If $x = u\tan v$, $y = u\sec v$ then prove that, $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial \cancel{y}}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$

We know that, $\sec^2 x - \tan^2 x = 1$

$$\therefore \frac{y^2}{u^2} \cancel{+} - \frac{x^2}{u^2} = 1$$

$$\therefore y^2 - x^2 = u^2$$

→ $$\left(\frac{\partial u}{\partial x}\right)_y = -2x = u\left(\frac{\partial u}{\partial x}\right)_y$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{u} \quad — (1)$$

→ $$x = u\tan v \quad \text{and} \quad y = \frac{u}{\cos v}$$

$$x = u\frac{\sin v}{\cos v} = \frac{\cancel{y}\cos v \cdot \sin v}{\cos v}$$

$$\therefore x = y \cdot \sin v$$

$$\therefore 1 = y\cos v\left(\frac{\partial v}{\partial x}\right)_y$$

$$\therefore \left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y\cos v} \quad — (2)$$

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Here, $y^2 - x^2 = u^2$

$\rightarrow$ $\left(\frac{\partial u}{\partial y}\right)_x = 2y = 2u\left(\frac{\partial u}{\partial y}\right)$

$\therefore \left(\frac{\partial u}{\partial y}\right) = \frac{y}{u}$ — (3)

$\rightarrow$ $x = u \tan v$ $\quad$ $y = u \sec v$

$u = \frac{x}{\tan v}$ $\quad$ $y = \frac{x \cdot \cos v}{\sin v} \cdot \frac{1}{\cos v}$

$\therefore y = \frac{x}{\sin v}$

$\therefore y \cdot \sin v = x$

$\therefore y \cdot \cos v \left(\frac{\partial u}{\partial y}\right)_x = x$

$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{x}{y \cos v}$ — (4)

$\rightarrow$ $\frac{x}{u} \times \frac{y}{y \cos v} = \frac{y}{u} \cdot \frac{x}{y \cos v}$

$\therefore \frac{x}{u \cos v} = \frac{x}{u \cos v}$ — (5)

therfore,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$$ is prove.

3 If $u = ax + by$ and $v = bx - ay$ then Find.

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{a^2}{a^2+b^2}$$ and

$$\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{a^2+b^2}{a^2}$$

-> $u = ax + by$ — ①

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = a \quad — ②$$

-> $v = bx - ay$

$\therefore ay = bx - v$

$$\therefore y = \frac{bx - v}{a}$$

put y value in eq^n 1

$$\therefore u = ax + b\left(\frac{bx - v}{a}\right)$$

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$\therefore Ua = a^2 x + b^2 x - bV$

derivative w.r.t. u

$\therefore a = a^2 + b^2 \left(\frac{\partial x}{\partial u}\right)_v$

$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2+b^2}$ — ③

→ We multiple eq^n 2 and 3

$\therefore \left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{a^2}{a^2+b^2}$

→ $V = bx - aY$

$\left(\cancel{\frac{\partial V}{\partial Y}}\right) 1 = -a \left(\frac{\partial y}{\partial v}\right)_x$

$\therefore \left(\frac{\partial Y}{\partial v}\right)_x = \frac{1}{-a}$ — ④

→ $V = bx - aY$

$\therefore bx = V + aY$

$\therefore x = \frac{V + aY}{b}$

put x value in eq^n 1

$$\therefore U = a\left(\frac{v+ay}{b}\right) + by$$

$$\therefore Ub = av + a^2y + b^2y$$

derivativie w.r.t to y,

$$\therefore 0 = a\left(\frac{\partial v}{\partial y}\right)_U + a^2 + b^2$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_U = -\frac{a^2+b^2}{a} \quad — \quad (5)$$

→ We multiple eq^n 4 and 5

$$\therefore \left(\frac{\partial y}{\partial u}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_U = \frac{a^2+b^2}{a^2}$$

4 If $x = r\cos\theta$ and $y = r\sin\theta$ then prove that,

$$\left[x\cdot\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = x^2 + y^2$$

→ $x = r\cos\theta$

$$\therefore \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

$\rightarrow \quad y = r \sin\theta$

$$\therefore \left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta$$

$$\rightarrow \text{L.H.S.} = \left[x \cdot \left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2$$

$$= \left[x(\cancel{x}\cos\theta) + y(\sin\theta)\right]^2$$

$$= \left[r\cos^2\theta + r\sin^2\theta\right]^2$$

$$= r^2$$

$$= x^2 + y^2$$

L.H.S. = R.H.S.

$$\left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = x^2 + y^2$$

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* Task 6 : Partial Differentiation of Composite Functions.

1 If $\tan^{-1}\left(\frac{x}{y}\right)$ and $x = 2t$ and $y = 1 - t^2$

then prove that $\frac{\partial z}{dt} = \frac{2}{1+t^2}$

$$\begin{tikzcd} & x \arrow[r, "\frac{dx}{dt}"] & t \\ z \arrow[ur, "\frac{\partial z}{\partial x}"] \arrow[dr, "\frac{\partial z}{\partial y}"'] & & \\ & y \arrow[r, "\frac{dy}{dt}"] & t \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} \cdot 2 + \frac{1}{1+\frac{x^2}{y^2}} \cdot \boxed{-\frac{x}{y^2}} (-2t)$$

$$= \frac{2y^2}{x^2+y^2} \cdot \frac{1}{y} + \frac{y^2}{x^2+y^2} \cdot \frac{2xt}{y^2}$$

$$= \frac{2y}{x^2+y^2} + \frac{2xt}{x^2+y^2}$$

$$= \frac{2(y+xt)}{x^2+y^2}$$

$$\frac{\partial z}{\partial t} = \frac{2\left(1 - t^2 + 2t(t)\right)}{4t^2 + 1 - 2t^2 + t^4}$$

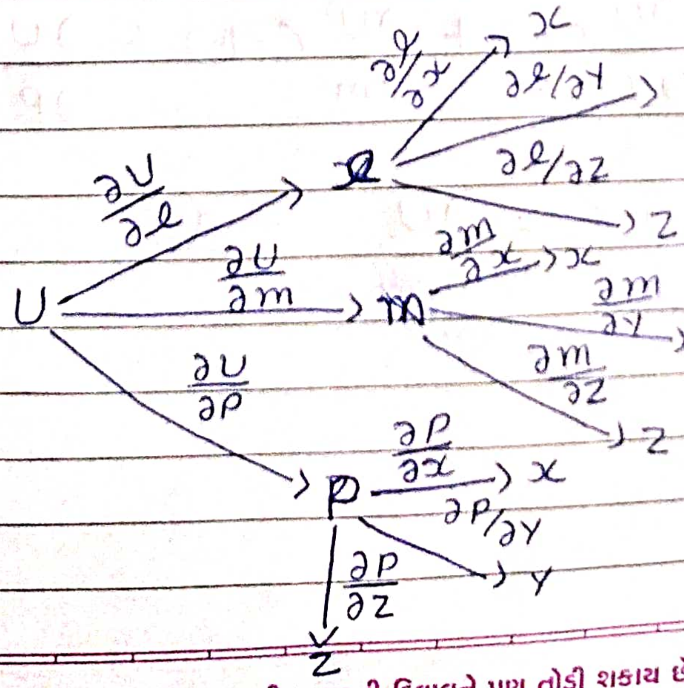
$$= \frac{2\left(1 - t^2 + 2t^2\right)}{2t^2 + 1 + t^4}$$

$$= \frac{2(1 + t^2)}{(1 + t^2)^2}$$

$$= \frac{2}{1 + t^2}$$

2 If $U = F(x - y,\ y - z,\ z - x)$ then prove that

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$ $x - y = l,\ y - z = m$, $z - x = p$



$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial x}$$

$$= \frac{\partial u}{\partial l}(1) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial p}(-1)$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial p} \quad \text{--- } (1)$$

$$\rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-1) + \frac{\partial u}{\partial m}(1) + \frac{\partial u}{\partial p}(0)$$

$$= \frac{\partial u}{\partial m} - \frac{\partial u}{\partial l} \quad \text{--- } (2)$$

$$\rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial z}$$

$$= \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial p}(1)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial m} \quad \text{--- } (3)$$

eq^n 1 + 2 + 3,

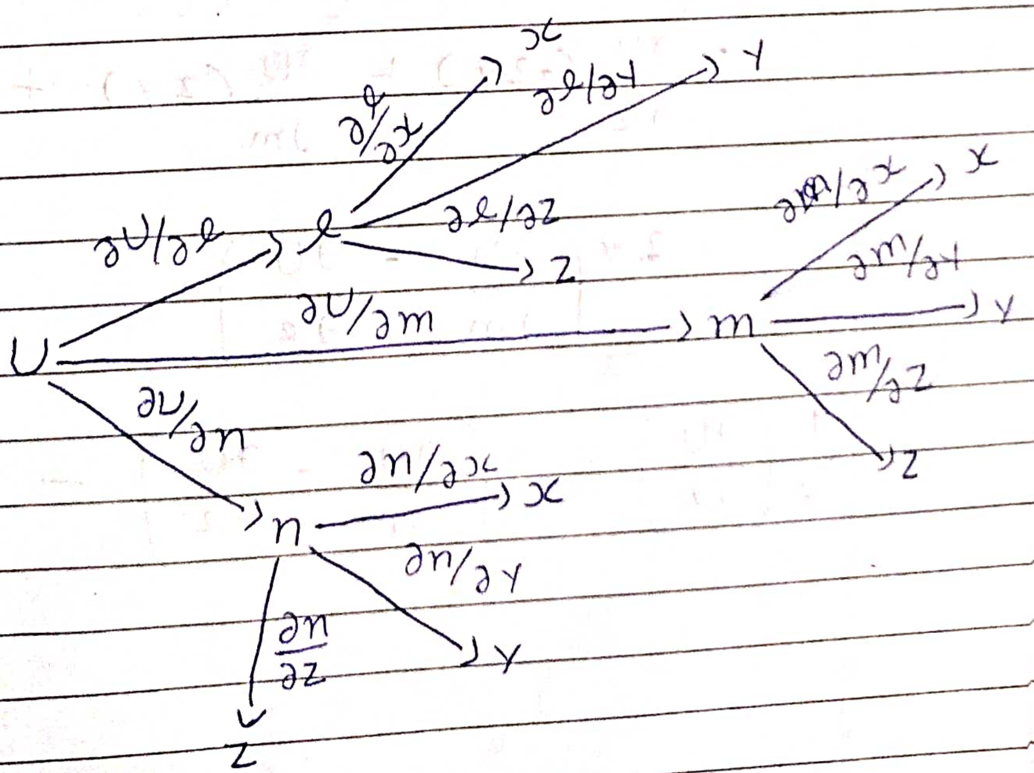
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial m} - \frac{\partial u}{\partial l} + \frac{\partial u}{\partial p} - \frac{\partial u}{\partial m}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

3z If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

We take, $x^2 - y^2 = l$
$y^2 - z^2 = m$
$z^2 - x^2 = n$



$\rightarrow$ $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l} \times 2x + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-2x)$$

$$= \frac{\partial u}{\partial l} 2x - 2x \cdot \frac{\partial u}{\partial n}$$

$$= 2x\left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right)$$

$$\frac{1}{x}\left(\frac{\partial u}{\partial x}\right) = 2\left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right) \quad \text{— } 1$$

$\rightarrow$ $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$= 2y\left(\frac{\partial u}{\partial m} - \frac{\partial u}{\partial l}\right)$$

$$\frac{1}{y}\left(\frac{\partial u}{\partial y}\right) = 2\left(\frac{\partial u}{\partial m} - \frac{\partial u}{\partial l}\right) \quad \text{— } 2$$

$$\rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial z}$$

$$= \frac{\partial U}{\partial l}(0) + \frac{\partial U}{\partial m}(-2z) + \frac{\partial U}{\partial n}(2z)$$

$$= 2z\left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m}\right)$$

$$\frac{1}{z}\left(\frac{\partial U}{\partial z}\right) = 2\left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m}\right) \quad — \quad (3)$$

$eq^n \; 1 + 2 + 3,$

$$= \frac{1}{x}\left(\frac{\partial U}{\partial x}\right) + \frac{1}{y}\left(\frac{\partial U}{\partial y}\right) + \frac{1}{z}\left(\frac{\partial U}{\partial z}\right)$$

$$= 2\left(\frac{\partial U}{\partial l} - \frac{\partial U}{\partial n}\right) + 2\left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial l}\right) + 2\left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m}\right)$$

$$= 0$$

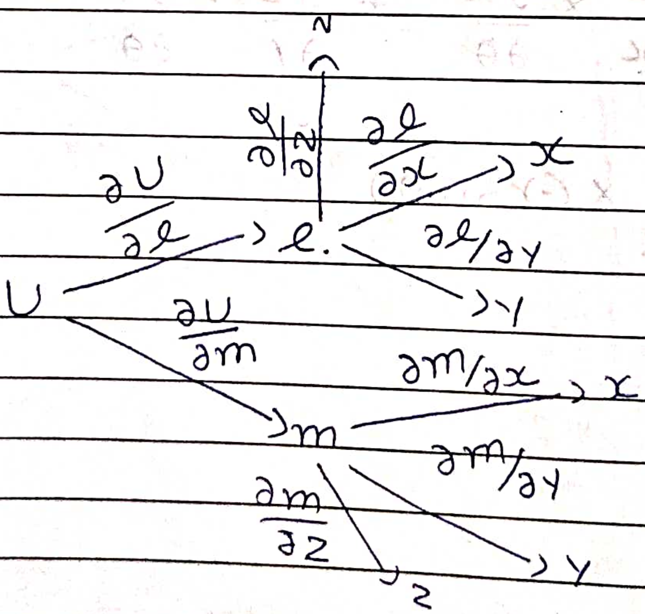
5 If $U = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ then prove

that, $x^2 \frac{\partial U}{\partial x} + y^2 \cdot \frac{\partial U}{\partial y} + z^2 \cdot \frac{\partial U}{\partial z} = 0$

Here we take $\frac{y-x}{xy} = l$, $\frac{z-x}{zx} = m$

$\Rightarrow l = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$

$\Rightarrow m = \frac{z}{zx} - \frac{x}{zx} = \frac{1}{x} - \frac{1}{z}$



$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x}$

$= \frac{\partial U}{\partial l}\left(\frac{-1}{x^2}\right) + \frac{\partial U}{\partial m}\left(\frac{-1}{x^2}\right)$

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial l} - \frac{\partial u}{\partial m} \quad \text{--- } (1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left[+\frac{1}{y^2}\right] + \frac{\partial u}{\partial m}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} \cdot y^2 = +\frac{\partial u}{\partial l} \quad \text{--- } (2)$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}\left[+\frac{1}{z^2}\right]$$

$$\Rightarrow z^2 \cdot \frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \quad \text{--- } (3)$$

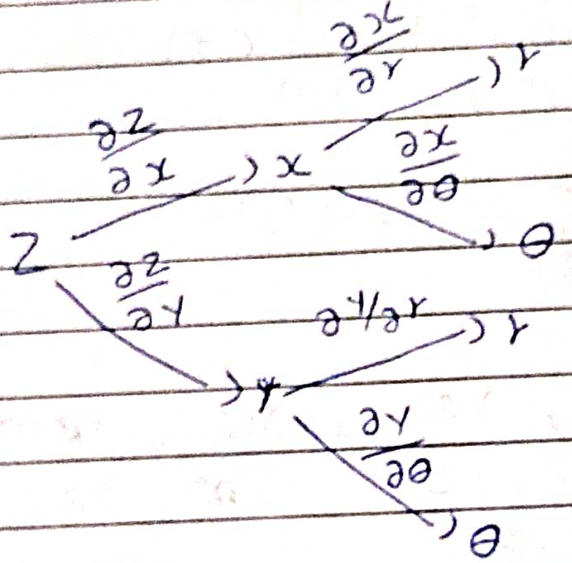
From eq^n 1 + 2 + 3,

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial l} - \frac{\partial u}{\partial m} + \frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}$$

$$\Rightarrow x^2 \cdot \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z} = 0$$

4 If $z = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$ then show that,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$



$$\Rightarrow \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad - (1)$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta \cdot \frac{\partial z}{\partial x} + r\cos\theta \cdot \frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{1}{r}\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial y}\cos\theta - \sin\theta \cdot \frac{\partial z}{\partial x} \quad - (2)$$

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$$L.H.S. = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \sin^2\theta \cdot \left(\frac{\partial z}{\partial y}\right)^2 + 2\sin\theta\cos\theta \cdot$$

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta$$

$$- 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \sin\theta \cdot \cos\theta$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$L.H.S. = R.H.S.$$

* Task 7: Euler's Theorem for Homogeneous Function

1 Check Function is Homogeneous or not

(i) $U = x^3 \sin\left(\frac{x}{y}\right)$

Replasing $x = xt$ and y by yt

$\therefore F(xt, yt) = x^3 t^3 \sin\left(\frac{xt}{yt}\right)$

$= x^3 t^3 \sin\left(\frac{x}{y}\right)$

$= t^3 (U)$

U is a homogeneous function of degree $n = 3$.

(ii) $U = \frac{x+y}{x^3+y^3}$

Replasing x by xt and y by yt

$\therefore F(xt, yt) = \frac{xt + yt}{x^3t^3 + y^3t^3}$

$= \frac{t(x+y)}{t^3 \; x^3+y^3}$

$F(xt, yt) = t^{-2}(u)$

U is a homoge. function of degree
$n = -2$.

(iii) $U = x^2y^3 + x^3y^2 + 7$

Replasing x by xt and y by yt

$F(xt, yt) = x^2t^2(y^3t^3) + x^3t^3(y^2t^2) + 7$

U is not homoge. function.

(iv) $U = \log x - \log y$

Replasing x by xt and y by yt.

$F(xt, yt) = \log(xt) - \log(yt)$

$= \log\left(\frac{xt}{yt}\right)$

$= \log\left(\frac{x}{y}\right)$

$= t^0(u)$

U is a homoge. function of degree
$n = 0$

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(v) $\sqrt[4]{\dfrac{x^5-y^5}{x-y}}$

Replasing x by xt and y by yt

$$F(xt, yt) = \sqrt[4]{\frac{x^5t^5-y^5t^5}{xt-yt}}$$

$$= t\left(\sqrt[4]{\frac{x^5-y^5}{x-y}}\right)$$

$$= t(u)$$

u is a homoge. function of degree $n=1$.

2 If $u = \left(\dfrac{x^2+y^2}{x+y}\right)$ then prove that

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = u$$

Let $F(u) = \dfrac{x^2+y^2}{x+y}$

Replasing x by xt and y by yt

$$F(xt, yt) = \frac{x^2t^2+y^2t^2}{xt+yt}$$

$$= f'\left(\frac{x^2+y^2}{x+y}\right)$$

$$= f'(U)$$

U is a homoge. function of degree
$n = 1$.

By Euler's theorem,

$$x.\frac{\partial U}{\partial x} + y.\frac{\partial U}{\partial y} = nU$$

$$= 1U$$

that prove, $x.\frac{\partial U}{\partial x} + y.\frac{\partial U}{\partial y} = U$

3 If $U = e^{x^2 f(x,y)}$, prove that

$$x.\frac{\partial U}{\partial x} + y.\frac{\partial U}{\partial y} = 2U.\log U$$

$$\therefore \log U = x^2 f(x,y).\log e$$

Replasing x by xt and y by yt

$$\therefore \log U = (x^2 t^2) f(xt, yt).\log e$$

$$\therefore f(xt, yt) = t^2 (x^2 f(x,y)).\log e$$

$= t^2 U$

U is a homoge. function of degree
$n = 2\log U$

By Euler's theorem,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$$

$$= 2U \cdot \log U$$

Heance prove that,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2U \cdot \log U$$

4 If $U = \sin^{-1}(x^2+y^2)^{1/5}$ then prove that,

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = \frac{2}{25} \tan U(2 \tan^2 U - 3)$$

$$U = \sin^{-1}(x^2+y^2)^{1/5}$$

$$\sin U = (x^2+y^2)^{1/5}$$

$$= (x^2t^2 + y^2t^2)^{1/5}$$

$$= t^{2/5} F(U)$$

$F(U) = \sin U$ is a homoge. function of degree $n = 2/5$

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$f(u) = \sin u$

$f'(u) = \cos u$

By Cor. 3

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2}$$

$$= g(u)[g'(u) - 1]$$

$$= \frac{f(u)}{f'(u)}[g'(u) - 1]$$

$$= \frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u - 1\right]$$

$$= \frac{2}{5}\tan u\left[\frac{2}{5} + \frac{2\tan^2 u}{5} - 1\right]$$

$$= \frac{2}{5}\tan u\left[\frac{2\tan^2 u - 3}{5}\right]$$

$$= \frac{2}{25}\tan u\left[2\tan^2 u - 3\right]$$

5 If $U = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$F(U) = \tan U = \frac{x^3+y^3}{x-y}$

$F'(U) = \sec^2 U$

Replasing x by xt and y by yt.

$F(xt, yt) = \frac{x^3t^3 + y^3t^3}{xt - yt}$

$= t^2(U)$

$F(U) = \tan U$ is homoge. function of degree $n=2$.

By Cro 3.

$$x^2 \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2}$$

$= g(U)[g'(U) - 1]$

$= \frac{F(U)}{F'(U)} [g'(U) - 1]$

$= \sin U \cos U [2\cos(2U) - 1 - 1]$

$= 2\,\sin U \cos U \cdot \cos 2U$

$= 2\cos(3U) \cdot \sin U$

Hence,

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial x \partial y} = 2\cos 3U \cdot \sin U$$

6 If $U = \dfrac{x^2 y^2 z^2}{x^2+y^2+z^2} + \cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)$

show that $x \cdot \dfrac{\partial U}{\partial x} + y \cdot \dfrac{\partial U}{\partial y} + z \cdot \dfrac{\partial U}{\partial z} = 4\,\dfrac{x^2+y^2+z^2}{x^2+y^2+z^2}$

Replasing x by xt, y by yt and z by zt

$$f(U) = \frac{x^2t^2 \cdot y^2t^2 \cdot z^2t^2}{x^2t^2 + y^2t^2 + z^2t^2} + \cos\left[\frac{x+(yt)+(yt)(zt)}{x^2t^2 + y^2t^2 + z^2t^2}\right]$$

$$= t^4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) + \cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$$

$$= t^4 U$$

U is a homo. function of degree $n = 4$

By Cor.

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = n \cdot U$$

$$= 4U$$

Hence,

prove that $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = 4U$

* Task : 8 Partial Differentiation of Implict function of two variable

2 If $x^3 + y^3 - 3axy = 0$ then Find $\frac{dy}{dx}$ at $(1, 1)$

Let $F(x, y) = x^3 + y^3 - 3axy$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= -\frac{(3x^2 - 3ay)}{(3y^2 - 3ax)}$$

$$= -\frac{x^2 + ay}{y^2 - ax}$$

$$\therefore \frac{dy}{dx}_{(1,1)} = \frac{-1 + 0}{1 - 0}$$

$$= -1$$

3 If $x^5 + y^5 - 5a^3x^2 = 0$ then Find $\frac{d^2y}{dx^2}$

Let $F(x, y) = x^5 + y^5 - 5a^3x^2$

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$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(5x^4 - 10xa^3)}{(5y^4)}$$

$$\therefore \frac{dy}{dx} = \frac{10xa^3 - 5x^4}{5y^4}$$

$$\therefore \frac{dy}{dx} = \frac{\cancel{10}2xa^3 - \cancel{5}x^4}{\cancel{5}y^4} = \frac{2xa^3 - x^4}{y^4}$$

$$\therefore \frac{dy}{dx} = \frac{2xa^3 - x^4}{y^4}$$

4 If $x \cdot e^y + \sin(xy) + y - \log 2 = 0$ then Find $\frac{dy}{dx}$ at $(0, \log 2)$

Let $F(x, y) = xe^y + \sin(xy) + y - \log 2$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-[e^y + y\cos(xy)]}{xe^y + \cos(xy) + 1}$$

$$\therefore \frac{dy}{dx} = \frac{-e^y - \cos(xy)y}{xe^y + \cos(xy) + 1}$$

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$$\frac{\partial u}{\partial x} = \frac{-e^{\log 2} - \log 2 (\cos 0)}{0 + 0 + 1}$$

$$\frac{\partial u}{\partial x} = -2 - \log 2$$

5 Find $\frac{dy}{dx}$ when $y^{x^y} = \sin x$

$$y^{x^y} = \sin x$$

$$\therefore x^y \cdot \log y = \log \sin x$$

Let $F(x, y) = x^y \log y - \log \sin x$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-[yx^{y-1} \log y - \cot x]}{x^y \cdot \log y \log x + x^y \cdot \frac{1}{y}}$$

$$= \frac{\cot x + yx^{y-1} \log y}{x^y \left(\log y \log x + \frac{1}{y}\right)}$$

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* Task 9: Tangent at Plane and normal line.

1 Find the equation of the tangent plane and normal line to the surface $xyz-6$ at $(1, 2, 3)$

Let, $f(x, y, z) = xyz-6$

$(x_0, y_0, z_0) = (1, 2, 3)$

$f_x(1, 2, 3) = yz = 6$

$f_y(1, 2, 3) = xz = 3$

$f_z(1, 2, 3) = xy = 2$

-> Tangent at plane.

$(x-x_0)f_x + (y-y_0)f_y + (z-z_0)f_z = 0$

$\therefore (x-1)6 + (y-2)3 + (z-3)2 = 0$

$\therefore 6x-6+3y-6+2z-6 = 0$

$\therefore 6x+3y+2z = 18$

-> Normal line,

$\therefore \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}$

2 Find the tangent plane to the surface $z = 1 - \frac{1}{10}(x^2 + 4y^2)$ at $(1, 1, \frac{1}{2})$

$$f(z) = 1 - \frac{1}{10}(x^2 + 4y^2)$$

$$(x_0, y_0, z_0) = (1, 1, 1/2)$$

$$f_x(1, 1, 1/2) = \frac{-2x}{10} = -1/5$$

$$f_y(1, 1, 1/2) = \frac{-8y}{10} = -4/5$$

$$f_z(1, 1, 1/2) = 0$$

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

$$\therefore 10z = 10 - x^2 - 4y^2$$

$$\therefore f(x, y, z) = x^2 + 4y^2 + 10z = 10$$

$$(x_0, y_0, z_0) = (1, 1, 1/2)$$

$$f_x(1, 1, 1/2) = 2x = 2$$

$$f_z(1, 1, 1/2) = 10$$

$$f_y(1, 1, 1/2) = 8y = 8$$

→ Tanget at Plane,

$$(x-1)2 + (y-1)8 + (z-\tfrac{1}{2})10 = 0$$

$$2x - 2 + 8y - 8 + \cancel{2z} - 10z - \cancel{10}\,5 = 0$$

$$2x + 8y + 10z = \cancel{20}\,15$$

$$\cancel{x + 4y - 5z = 10}$$

→ Normal line

$$\therefore \frac{(x-1)}{2} = \frac{y-1}{8} = \frac{2z-1}{20}$$

3 Find the equation of the tangent plane and normal line to the surface $z+8 = xe^{y}\cos z$ at point $(8,0,0)$

$$\therefore xe^{y}\cos z - z = 8$$

Let $f(x,y,z) = xe^{y}\cos z - z - 8 = 0$

$(x_0, y_0, z_0) = (8,0,0)$

⇒ $f_x = e^{y}\cos z = 1$

$f_y = xe^{y}\cos z = 8$

$f_z = \sin(z) - 1 = -1$

=> Tangent at Plane.

$(x-8)1+(y-0)8+(-1)(z)=0$

$\therefore x-8+8y-z=0$

$\therefore x+8y-z=8$

=> Normal line is,

$\therefore \frac{x-8}{1}=\frac{y-0}{8}=\frac{z}{(-1)}$

$\therefore x-8=\frac{y}{8}=(-z)$

4 Show that plane $3x+12y-6z-17=0$ touches the surface $3x^2-6y^2+9z^2+17=0$ Find point.

Point $=P(x_1, y_1, z_1)$

$f(x,y,z)=3x^2-6y^2+9z^2+17$

$\frac{\partial f}{\partial x}=6x=6x_1$

$f_y=-12y=-12y_1$

$f_z=-18z=-18z_1$

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$\Rightarrow$ Tangent at plan

$\therefore (x - x_1)(6x_1) + (y - y_1)(-12y_1) + (z - z_1)(18z_1) = 0$

$\therefore 6xx_1 - 6x_1^2 + 12yy_1 + 12y_1^2 + 18zz_1 - 18z_1^2 = 0$

$\therefore 3xx_1 - 6yy_1 + 9zz_1 + 17 = 0$

$\Rightarrow$ Comparing with $3x + 12y - 6z - 17 = 0$

$\therefore \frac{x_1}{1} = \frac{-y_1}{2}, = \frac{3z_1}{-2} = -1$

$\therefore P(x_1, y_1, z_1) = (-1, 2, 2/3)$

Task : 10 : Jacobin

1 Find the Jacobian $J\left(\frac{u, v}{x, y}\right)$ for

$u = e^{x} \sin y$, $v = x \log(\sin y)$

$$\Rightarrow J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^{x} \sin y & e^{x} \cos y \\ \log \sin y & \frac{x \cos y}{\sin y} \end{vmatrix}$$

$$= \begin{vmatrix} e^{x} \sin y & e^{x} \cos y \\ \log(\sin y) & x \cot y \end{vmatrix}$$

$$= \left(e^{x} . x \sin y . \cot y - e^{x} \cos y . \log(\sin y)\right)$$

$$= e^{x} . x \cos y - e^{x} \cos y . \log \sin y$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = e^{x} \cos y \left[x - \log(\sin y)\right]$$

2 Calculate $\frac{\partial(u,v)}{\partial(x,y)}$ for $x = e^u \cos v$
$y = e^u \sin v$

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$J =$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} e^u \cos v & -e^u \sin v \\ +e^u \sin v & e^u \cos v \end{vmatrix}$

$= e^{2u}(\cos^2 v + \sin^2 v)$

$\frac{\partial(x,y)}{\partial(u,v)} = e^{2u} = J^*$

$J * J^* = 1$

$J = \frac{1}{J^*} = \frac{1}{e^{2u}} = e^{-2u} = \frac{\partial(u,v)}{\partial(x,y)}$

3 If $U = xyz$, $V = x^2 + y^2 + z^2$, $W = x + y + z$ then Find $\dfrac{\partial(x, y, z)}{\partial(U, V, W)}$

$$J^* = \frac{\partial(U, V, W)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - xz(2x - 2z) + xy(2x - 2y)$$

$$= 2y^2z - 2z^2y - 2x^2z + 2z^2x$$

$$+ 2x^2y - 2y^2x$$

$$\cancel{= 2(y^2z - z^2y) + (z^2x - x^2z) + x^2y}$$

$$= 2\left(zy^2 - z^2y - x^2z + z^2x + x^2y - xy^2\right)$$

$$= -2(x - y)(y - x)(z - x)$$

4 If $x = r\sin\theta.\cos\phi$, $y = r\sin\theta\sin\phi$
$z = r\cos\theta$ then show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cdot\cos\phi & r\cos\theta.\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta.\sin\phi & r\cos\theta.\sin\phi & r\cos\phi\cdot\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta.\cos\phi\,(r^2\sin^2\theta\cdot\cos\phi) -$$

$$r\cos\theta.\cos\phi\,(-r\cos\theta.\cos\phi\sin\theta)$$

$$+(-r\sin\theta.\sin\phi)\left(\frac{-r\sin^2\theta}{\sin\phi} - \frac{r\cos^2\theta}{\sin\phi}\right)$$

$$= r^2\sin^3\theta.\cos^2\phi + r^2\cos^2\theta\cos^2\phi\sin\theta$$

$$+ (r^2\sin^3\theta\sin^2\phi + \frac{r^2\cos^2\theta\sin^2\theta}{\sin\theta}$$

$$= r^2\sin\theta$$

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5 If $x = u$, $y = u\tan v$, $z = w$ then verify the $JJ' = 1$

$\Rightarrow$ $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = 1(u\sec^2 v) = u\sec^2 v \quad - (1)$$

$\Rightarrow$ $J^* = \dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

$$J^* = \begin{vmatrix} 1 & 0 & 0 \\ \frac{x^2-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{u}{u^2(1+\tan^2 v)}$$

$$= \frac{1}{u\sec^2 v} \quad - (2)$$

$\Rightarrow$ L.H.S. $= J * J^*$

$$= u\sec^2 v \cdot \frac{1}{u\sec^2 v}$$

$= 1 = J * J^*$ is prove.

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* Task 11 : Taylor's Theorem for the Function of two variable

1 Expand $F(x,y) = x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$

Let $F(x,y) = x^2y + 3y - 2 = (1)^2(-2)^2 + 3(-2) - 2$

$F(a,b) = -10$

=) Taylor's Series,

$$F(x,y) = F(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)$$

$$f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \dots$$

=) Here, $x = 1$, $y = -2$

=) $f_x = 2xy = 2(-2) = -4$

=) $f_y = x^2 + 3 = 4$

=) $f_{xx} = 2$ =) $f_{xy} = 2x = 2$

=) $f_{yy} = 0$

=) $F(x,y) = -10 + [(x-1)(-4) + (y+2)4] +$

$$1\,[(x-1)^2(2) + 2(x-1)(y-2) + 0]$$

2 Expand $e^x \cos y$ in power of x and $(y-\frac{\pi}{4})$ upto second degree.

$\Rightarrow$ Let $F(x,y) = e^x \cos y$

$\Rightarrow$ Here $a=0$ and $b = \pi/4$

$\Rightarrow$ Taylor's Series,

$$F(x,y) = F(a,b) + \left[(x-a)F_x + (y-b)F_y\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 F_{xx} + 2(x-a)(y-b)F_{xy} + (y-b)^2 F_{yy}\right] + \ldots$$

$\rightarrow$ $F_x = e^x \cos y = \frac{1}{\sqrt{2}}$

$\rightarrow$ $F_y = -e^x \sin y = \frac{-1}{\sqrt{2}}$

$\rightarrow$ $F_{xx} = e^x \cos y = \frac{1}{\sqrt{2}}$

$\rightarrow$ $F_{yy} = -e^x \cos y = \frac{-1}{\sqrt{2}}$

$\rightarrow$ $F_{xy} = -e^x \sin y = \frac{-1}{\sqrt{2}}$

$$\Rightarrow e^{x}\cos y = \frac{1}{\sqrt{2}} + \left(\frac{x}{\sqrt{2}} - \frac{(y-\frac{\pi}{4})}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{x^2}{\sqrt{2}} - \frac{2x(y-\pi/4)}{\sqrt{2}} - \frac{(y-\pi/4)^2}{\sqrt{2}}\right) + \cdots$$

3 Expand $\cos x \cos y$ in Power of x and y upto third degree terms.

Here $a = 0$ and $b = 0$

$\Rightarrow$ Taylor's Serie.

$$F(x,y) = F(a,b) + \left((x) f_x + y f_y\right) + \frac{1}{2!}\left(x^2 f_{xx} + 2xy \ldots + y^2 f_{yy}\right) + \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{x^2y} + 3x^2 x f_{y^2x} + y^3 f_{yyy}\right) + \cdots$$

$\rightarrow F_x = -\sin x \cos y = 0$

$\rightarrow F_y = -\cos x \sin y = 0$

$\rightarrow F_{xx} = -\cos x \cos y = -1$

$\rightarrow F_{xxx} = \sin x \cos y = 0$

$\rightarrow \quad F_{yy} = -\cos x \cos y = -1$

$\rightarrow \quad F_{yyy} = +\cos x \sin y = 0$

$\rightarrow \quad F_{x^2y} = \cos x \sin y = 0$

$\rightarrow \quad F_{y^2x} = \sin x \cos y = 0$

$$\Rightarrow \cos x \cos y = 1 + [x(0) + y(0)] + \frac{1}{2!}[x^2(-1) + 2xy + y^2(-1)]$$

$$+ \frac{1}{3!}(0) + \cdots$$

$$\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \cdots$$

4 Expand $\tan^{-1}(y/x)$ about $(1,1)$.

Here $a=1$, $b=1$

$\rightarrow \quad F(a,b) = \pi/4$

$$\rightarrow \quad F_x = \frac{1}{1 + y^2/x^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-1}{2}$$

$$\rightarrow \quad F_{xx} = \frac{-y(2x)}{(x^2+y^2)^2} = \frac{1}{2}$$

$\rightarrow F_y = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x} = \dfrac{1}{2}$

$\rightarrow F_{yy} = \dfrac{1 - x(2y)}{(x^2+y^2)} = \dfrac{-1}{2}$

$\rightarrow F_{xy} = \dfrac{-[x^2+y^2-2y^2]}{(x^2+y^2)^2} = 0$

$\rightarrow F(x,y) = \dfrac{\pi}{4} + \left[(x-1)\left(-\dfrac{1}{2}\right) + (y-1)\left(\dfrac{1}{2}\right)\right]$

$+ \dfrac{1}{2!}\left[\dfrac{(x-1)^2}{4} \cancel{+} - \dfrac{(y-1)^2}{4}\right] + \dots$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}(x-1) + \dfrac{1}{2}(y-1) + \dfrac{1}{4}(x-1)^2$

$- \dfrac{1}{4}(y-1)^2 + \dots$

* Task 12: Maxima, Minima For the Function of two variables.

1 Find the extreme values of $x^2+y^2+xy+x-4y+5$.

$\Rightarrow$ Let $f(x,y) = x^2+y^2+xy+x-4y+5$

$$\frac{\partial f}{\partial x} = 2x+y+1 = 0$$

$$\frac{\partial f}{\partial y} = 2y+x-4 = 0$$

$\therefore 2y = 4-x$

$\therefore y = \frac{4-x}{2}$

$\therefore 2x + \frac{4-x}{2} + 1 = 0$

$\therefore 4x+4-x+2 = 0$

$\therefore 3x+6 = 0$

$\therefore x = -2 \rightarrow y = \frac{4+2}{2} = 3$

point is $(-2, 3)$

$\Rightarrow$ $r = \frac{\partial^2 f}{\partial x^2} = 2$, $t = \frac{\partial^2 f}{\partial y^2} = 2$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$\Rightarrow rt - s^2 = 4 - 1 = 3 > 0$$

$$r = 2 > 0$$

Hence, $F(x, y)$ is minimum at $(-2, 3)$ point

$$f_{min} = (-2)^2 + (3)^2 + (-2)(3) - 2 - 4(3) + 5$$

$$= 4 + 9 - 6 - 2 - 12 + 5$$

$$f_{min} = -2$$

2 Find extreme values of $x^3 + 3xy^2 - 15y^2 + 72x$

$\Rightarrow$ Let $f(x, y) = x^3 + 3xy^2 - 15x^2 + 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 + 72 = 0$$
$$+30x^2$$

$$\therefore \cancel{x^2 + y^2 + 24 = 0}$$

$$\frac{\partial f}{\partial y} = 6xy - 30y = 0$$

$$\therefore xy - 5y = 0$$

$$\therefore y(x - 5) = 0$$

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$\Rightarrow x = 5, \quad y = 0$

$\rightarrow$ For $x = 5, \rightarrow 3(5)^2 + 3(y^2) + 72 + 3d(5) = 0$

$\therefore \quad y = \pm 2$

$\rightarrow$ For $y = 0$

$\therefore \quad 3x^2 + 3y^2 - 30x - 72 = 0$

$\therefore \quad 3x^2 - 30x + 72 = 0$

$\therefore \quad x = 6, \quad x = 4$

$\rightarrow$ ponits are $(6,0), (4,0) (5,+2), (5,-2)$

$\Rightarrow r = \dfrac{\partial^2 f}{\partial x^2} = 6x - 30$

$\Rightarrow s = \dfrac{\partial^2 f}{\partial x \partial y} = 6y$

$\Rightarrow t = \dfrac{\partial^2 f}{\partial y^2} = 6x - 30$

$\Rightarrow rt - s^2 = (6x-30)(6x-30) - 36y^2$

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$\Rightarrow$	(x, y)	r	s	t	$rt - s^2$	Conclusion
1	$(4, 0)$	$-6 < 0$	0	-6	$36 > 0$	maximum
2	$(6, 0)$	$6 > 0$	0	6	$36 > 0$	minimum
3	$(5, 2)$	0	6	0	$-36 < 0$	neither max. or min.
4	$(5, 2)$	0	-6	0	$-36 < 0$	neither max or min

Hence, $F(x, y)$ is maximum at $(4, 0)$ and minimum at $(6, 0)$

$F_{max} = 112$

$F_{min} = 108$

3 Find the extreme values of $f(x, y) = \sin x + \sin y + \sin(x+y)$

$\Rightarrow \dfrac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0$

$\Rightarrow \dfrac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0$

$\therefore \cos x = \cos y$

$\therefore x = y$

$\Rightarrow \cos x + \cos 2x = 0$

$\therefore \cos x = -\cos 2x$

$\therefore x = \pi - 2x$

$\therefore x = \pi/3$

$\Rightarrow \cos x = \cos(\pi + 2x)$

$\therefore x = \pi + 2x$

$\therefore x = -\pi$

$\Rightarrow \cos y + \cos 2y = 0$

$\therefore y = \pi/3, \ -\pi$

$\Rightarrow$ point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (-\pi, -\pi)$

$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$\Rightarrow s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$\Rightarrow t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

	(x, y)	r	s	t	$rt - s^2$	Conclusion
1	$(\frac{\pi}{3}, \frac{\pi}{3})$	$-\sqrt{3}<0$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$9/4 > 0$	maximum
2	$(-\pi, -\pi)$	0	0	0	0	no conclusion

$\Rightarrow F_{max} = \frac{3\sqrt{3}}{2}$

5 Determine the points where the function $x^3 + y^3 - 3axy$ has a maximum or minimum.

$\Rightarrow$ Let $F(x, y) = x^3 + y^3 - 3axy$

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3ay = 0 = x^2 - ay$

$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0 = y^2 - ax$

$\therefore \cancel{y^2 = ax}$

$\therefore$ putting $y = \frac{x^2}{a}$

$\therefore x^4 - a^3 x = 0$

$\therefore x(x-a)(x^2 + ax + a^2) = 0$

$\therefore x = 0 \; ; \; x = a$

$y = 0 \; ; \; y = a$

$\Rightarrow$ point $(0,0)$ and (a,a)

$\Rightarrow$ $r = \frac{\partial^2 f}{\partial x^2} = 2x$

$\Rightarrow$ $s = \frac{\partial^2 f}{\partial x \partial y} = -a$

$\Rightarrow$ $t = \frac{\partial^2 f}{\partial y^2} = 2y$

$\Rightarrow$ For (a,a) point

$r > 0, \quad 3a^2 > 0$

Hense, $f(x,y)$ is minimum at (a,a)

$f_{min} = -a^3$

4 A rectangular box open at the top is to have a volume 108 cubic meter. what must be the ~~de~~ dimension so that the total surface area is minimum