

Chapter - 4 Beta and Gamma Function

* Task - 1 Evaluate the following integrals

1 $\int_{-\infty}^{\infty} e^{-4x^2} \cdot dx$

take $4x^2 = t \quad \rightarrow x = \frac{\sqrt{t}}{2}$
 $\therefore 8x \cdot dx = dt$

$\therefore dx = \frac{dt}{8x} = \frac{\sqrt{t} \cdot dt}{4}$

$$= \int_{-\infty}^{\infty} e^{-t} \cdot \frac{\sqrt{t}}{4} \cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-t} \cdot \frac{t^{\frac{1}{2}}}{4} \cdot dt$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} e^{-t} \cdot t^{1-\frac{1}{2}} \cdot dt$$

$$= \frac{\Gamma(1/2)}{4}$$

$$= \frac{\sqrt{\pi}}{4}$$

$$2 \int_0^{\infty} x^2 \cdot e^{-4x^4} \cdot dx$$

$$\text{take } x^4 = t$$

$$\therefore 4x^3 \cdot dx = dt$$

$$\therefore dx = \frac{dt}{4x^3}$$

$$\therefore x = \sqrt[4]{t}$$

$$\therefore x^2 = t^{1/2}$$

$$\therefore x^3 = t^{3/4}$$

$$= \int_0^{\infty} x^2 \cdot e^{-t} \cdot \frac{dt}{4x^3}$$

$$\therefore \int_0^{\infty} \frac{e^{-t} \cdot dt}{4 \cdot t^{1/4}}$$

$$\therefore \frac{1}{4} \int_0^{\infty} e^{-t} \cdot t^{-1/4} \cdot dt$$

$$= \frac{1}{4} \int_0^{\infty} e^{-t} \cdot t^{\frac{3}{4} - 1} \cdot dt$$

$$= \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$3 \int_0^1 \sqrt{x} (1-x^2)^{\frac{1}{3}} \cdot dx$$

$$\text{take } x^2 = t \quad \rightarrow x = \sqrt{t}$$

$$\therefore 2x \cdot dx = dt$$

$$\therefore dx = \frac{dt}{2x}$$

$$\therefore dx = \frac{dt}{2\sqrt{t}}$$

$$= \int_0^1 t^{\frac{1}{4}} (1-t)^{\frac{1}{3}} \cdot \frac{dt}{2t^{\frac{1}{2}}}$$

$$= \frac{1}{2} \int_0^1 t^{-\frac{1}{4}} \cdot (1-t)^{\frac{1}{3}} \cdot dt$$

$$= \frac{1}{2} \int_0^1 t^{\frac{3}{4}-1} \cdot (1-t)^{\frac{4}{3}-1} \cdot dt$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{4}{3}\right)$$

$$4 \int_0^2 x^4 (8-x^3)^{-\frac{1}{3}} \cdot dx$$

$$\text{take } x^3 = 8t$$

$$\therefore 3x^2 \cdot dx = 8 \cdot dt$$

$$\therefore dx = \frac{8}{3} \cdot \frac{dt}{x^2}$$

$$\therefore dC = \frac{dt \times 8}{3 \cdot t^{2/3}}$$

$$= \int_0^1 16 \cdot t^{4/3} \cdot (8 - 8t)^{-1/3} \cdot \frac{dt}{6}$$

$$= \int_0^1 \frac{16 \times 2}{6} \cdot t^{4/3} \cdot (1-t)^{-1/3} \cdot dt$$

$$= \frac{16}{3} \int_0^1 (1-t)^{\frac{2}{3}-1} \cdot t^{\frac{5}{3}-1} \cdot dt$$

$$= \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$$

* Task-2 Evaluate the following integrals.

$$1 \int_0^{\pi} \sin^2 x (1 + \cos x)^4 \cdot dx$$

$$= \int_0^{\pi} \sin^2 x \left(2 \cos^2 \frac{x}{2}\right)^4 \cdot dx$$

$$= 16 \int_0^{\pi} \sin^2 x \cdot \cos^8 \frac{x}{2} \cdot dx$$

Here, we take $\frac{x}{2} = t$

$$\therefore \frac{1}{2} \cdot dx = dt$$

$$\therefore dx = 2 \cdot dt$$

$$\therefore x = 2t$$

When $x = \pi$, $\frac{x}{2} \rightarrow \frac{\pi}{2}$
 $x = 0$, $\frac{x}{2} \rightarrow 0$

$$= 16 \int_0^{\pi/2} \sin^2 2t \cdot \cos^8 x \cdot 2 \cdot dt$$

$$= 16 \int_0^{\pi/2} 2 \sin^2 t \cdot \cos^2 t \cdot \cos^8 x \cdot 2 \cdot dt$$

$$= 64 \int_0^{\pi/2} \sin \theta \cdot \cos^9 \theta \cdot d\theta$$

$$= 64 \cdot \frac{1}{2} B \left(\frac{1+1}{2}, \frac{9+1}{2} \right) \cdot d\theta$$

$$= 32 \frac{|1| |5|}{|1+5|}$$

$$= 32 \int_0^{\pi/2}$$

$$= 64 \int_0^{\pi/2} 2 \sin^2 \theta \cdot \cos^{10} \theta \cdot d\theta$$

$$= 128 \int_0^{\pi/2} \sin^2 \theta \cdot \cos^{10} \theta \cdot d\theta$$

$$= 128 \frac{B \left(\frac{2+1}{2}, \frac{10+1}{2} \right)}$$

$$= 64 \frac{|3/2| \cdot |11/2|}{|\frac{3}{2} + \frac{11}{2}|}$$

$$= 64 \times \frac{1}{2} \times \frac{1}{7} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{21\pi}{16}$$

$$2 \int_0^{\pi/2} \sin 3x \cdot \cos^5 x \cdot dx$$

$$= \int_0^{\pi/2} (3\sin x - 4\sin^3 x) \cdot \cos^5 x \cdot dx$$

$$= \int_0^{\pi/2} 3\sin x \cdot \cos^5 x \cdot dx - 4 \int_0^{\pi/2} \sin^3 x \cdot \cos^5 x \cdot dx$$

$$= 3 \frac{\beta}{2} \left(\frac{1+1}{2}, \frac{5+1}{2} \right) - 4 \frac{\beta}{2} \left(\frac{3+1}{2}, \frac{5+1}{2} \right)$$

$$= \frac{3}{2} \frac{1 \cdot 3}{1+3} - \frac{4}{2} \frac{2 \cdot 3}{2+3}$$

$$\ll = \frac{3 \times 2 \times 1}{2 \times 3 \times 2} - \frac{2 \times 2 \times 1}{4 \times 3 \times 2}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3}$$

$$3 \int_0^1 \frac{x^5 \cdot \sin^{-1} x \cdot dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x \int_0^1 x^5 \cdot dx - \int_0^1 \frac{d}{dx} \sin^{-1} x \cdot \int_0^1 x^5 \cdot dx$$

$$= \left[\sin^{-1} x \left[\frac{x^6}{6} \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^6}{6} \cdot dx \right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{6} - \frac{1}{6} \int_0^1 \frac{1 \cdot x^6}{\sqrt{1-x^2}} \cdot dx$$

Here we take $x = \sin \theta$

$$\therefore dx = \cos \theta \cdot d\theta$$

x	0	1
θ	0	$\frac{\pi}{2}$

$$= \frac{\pi}{12} - \frac{1}{6} \int_0^{\frac{\pi}{2}} \frac{\sin^6 \theta \cdot \cos \theta \cdot d\theta}{\sqrt{1-\sin^2 \theta}}$$

$$= \frac{\pi}{12} - \frac{1}{6} \int_0^{\frac{\pi}{2}} \frac{\sin^6 \theta \cdot \cos \theta \cdot d\theta}{\cos \theta}$$

$$= \frac{\pi}{12} - \frac{1}{6} \int_0^{\frac{\pi}{2}} \sin^6 \theta \cdot \cos^0 \theta \cdot d\theta$$

$$= \frac{\pi}{12} - 6 \frac{\beta}{2} \left(\frac{6+1}{2}, \frac{1}{2} \right)$$

$$= \frac{\pi}{12} - \frac{3 \sqrt{7/2} \cdot \sqrt{1/2}}{\sqrt{7/2 + 1/2}}$$

$$= \frac{\pi}{12} - \frac{3 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{1/2} \times \sqrt{1/2}}{\sqrt{4}}$$

$$= \frac{\pi}{12} - \frac{5\pi}{32 \times 6}$$

$$= \frac{11\pi}{192}$$

4 $\int_0^1 \frac{x^7}{\sqrt{1-x^2}} dx$

Here we take $x = \sin \theta$

$$\therefore dx = \cos \theta \cdot d\theta$$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{\sin^7 \theta}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^7 \theta \cdot \cos \theta \cdot d\theta}{\cos \theta}$$

$$= \int_0^{\pi/2} \sin^7 \theta \cdot \cos \theta \cdot d\theta$$

$$= \frac{\beta}{2} \left(\frac{7+1}{2}, \frac{1}{2} \right)$$

$$= \frac{\beta}{2} \left(4, \frac{1}{2} \right)$$

$$= \frac{1}{2} \frac{|4| \sqrt{1/2}}{\sqrt{4 + \frac{1}{2}}}$$

$$= \frac{1 \times 2 \times \sqrt{1/2}}{2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{1/2}}$$

$$= \frac{16}{35}$$

$$\int_0^{\pi} (1 - \cos \theta)^5 \cdot d\theta$$

We know that, $1 - \cos \theta = \frac{2 \sin^2 \theta}{2}$

$$= \int_0^{\pi} \left(\frac{2 \sin^2 \theta}{2} \right)^5 \cdot d\theta$$

$$= 32 \int_0^{\pi} \sin^2 \frac{\theta}{2} \cdot d\theta$$

Here we take $\frac{\theta}{2} = t$

$$\therefore d\theta = 2 \cdot dt$$

θ	0	π
t	0	$\theta\pi/2$

$$= 32 \int_0^{\pi/2} \sin^2 t \cdot 2 \cdot dt$$

$$= 64 \cdot \frac{13}{2} \left(\frac{10+1}{2}, \frac{1}{2} \right)$$

$$= 32 \cdot \frac{11/2 \cdot 1/2}{\frac{11}{2} + \frac{1}{2}}$$

$$= \frac{32 \times 9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times 1/2 \times 1/2}{5 \times 4 \times 3 \times 2}$$

$$= \frac{63\pi}{8}$$

$$6 \int_0^{\infty} \frac{1}{(1+x^2)^{9/2}} \cdot dx$$

Here we take $x = \tan \theta$

$$\therefore d\theta = \sec^2 \theta \cdot d\theta$$

$$\therefore$$

x	θ	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{1}{(1+\tan^2 \theta)^{9/2}} \cdot \sec^2 \theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{1}{(\sec^2 \theta)^{9/2}} \cdot \sec^2 \theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2 \theta \cdot d\theta}{\sec^9 \theta}$$

$$= \int_0^{\pi/2} \frac{1}{\sec^7 \theta} \cdot d\theta$$

$$= \int_0^{\pi/2} \cos^7 \theta \cdot d\theta$$

$$= \frac{\beta}{2} \left(\frac{1}{2}, \frac{7+1}{2} \right)$$

$$= \frac{\beta}{2} \left(\frac{1}{2}, 4 \right)$$

$$= \frac{1}{2} \frac{\left[\frac{1}{2} \mid 4 \right]}{\left[\frac{1}{2} + 4 \right]}$$

$$= \frac{1}{2} \cdot \frac{\left[\frac{1}{2} \times 2 \right]}{2 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\frac{1}{2}}}$$

$$= \frac{16}{35}$$

$$7 \int_0^{\infty} \frac{1}{(1+x^2)^5} \cdot dx$$

Here we take $x = \tan \theta$

$$\therefore dx = \sec^2 \theta \cdot d\theta$$

x	0	∞
θ	0	$\frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{(1+\tan^2 \theta)^5} \cdot \sec^2 \theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2 \theta}{\sec^{10} \theta} \cdot d\theta$$

$$= \int_0^{\pi/2} \cos^8 \theta \cdot d\theta$$

$$= \frac{\beta}{2} \left(\frac{1+0}{2}, \frac{8+1}{2} \right)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{35\pi}{256}$$

$$\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}} \cdot dx$$

Here we take $x^3 = \tan \theta$

$$\therefore 3x^2 \cdot dx = \sec^2 \theta \cdot d\theta$$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{(\tan^2 \theta)^{1/3}}{(1+\tan^2 \theta)^{7/2}} \cdot \frac{\sec^2 \theta}{3(\tan^2 \theta)^{1/3}} \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} \cos^5 \theta \cdot d\theta$$

$$= \frac{1}{3} \cdot \frac{\beta}{2} \left(\frac{1}{2}, \frac{5+1}{2} \right)$$

$$= \frac{1}{6} \times \frac{1/2 \cdot 3}{\sqrt{1/2 + 3}}$$

$$= \frac{1}{6} \cdot \frac{1/2 \cdot 2}{6 \times 5/2 \times 3/2 \times 1/2 \times 1/2}$$

$$= \frac{8}{45}$$

9 $\int_0^{\pi} (1 + \cos \theta)^3 \cdot d\theta$

We know that $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$

$$= \int_0^{\pi} 8 \cos^6 \frac{\theta}{2} \cdot d\theta$$

Here we take $\frac{\theta}{2} = t$

$$\therefore d\theta = 2 \cdot dt$$

θ	0	π
t	0	$\pi/2$

$$= 8 \int_0^{\pi/2} \cos^6 t \cdot 2 \cdot dt$$

$$= 16 \int_0^{\pi/2} \cos^6 t \cdot dt$$

$$= 16 \cdot \frac{\pi}{2} \left(\frac{1}{2}, \frac{6+1}{2} \right)$$

$$= 8 \cdot \frac{1/2 \cdot 7/2}{\frac{7}{2} + \frac{7}{2}}$$

$$= 8 \cdot \frac{1/2 \cdot 5/2 \times 3/2 \times 1/2 \times 1/2}{3 \times 2 \times 1} \cdot \frac{7}{2}$$

$$= \frac{5\pi}{2}$$

$$10 \int_0^{\pi/4} (\cos 2\theta) \cdot d\theta$$

Here we take $2\theta = t$
 $\therefore 2 \cdot d\theta = dt$
 $\therefore d\theta = \frac{dt}{2}$

θ	0	$\pi/4$
t	0	$\pi/2$

$$\frac{\pi}{2}$$
$$= \int_0^{\frac{\pi}{2}} \cos^7 \theta \cdot \frac{d\theta}{2}$$

$$= \frac{1}{2} \cdot \frac{\beta}{2} \left(\frac{1}{2}, \frac{7+1}{2} \right)$$

$$= \frac{1}{4} \cdot \frac{\sqrt{1/2} \sqrt{4}}{\sqrt{1/2 + 4}}$$

$$= \frac{1 \times \sqrt{1/2 \times 3 \times 2}}{4 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{1/2}}$$

$$= \frac{8}{35}$$