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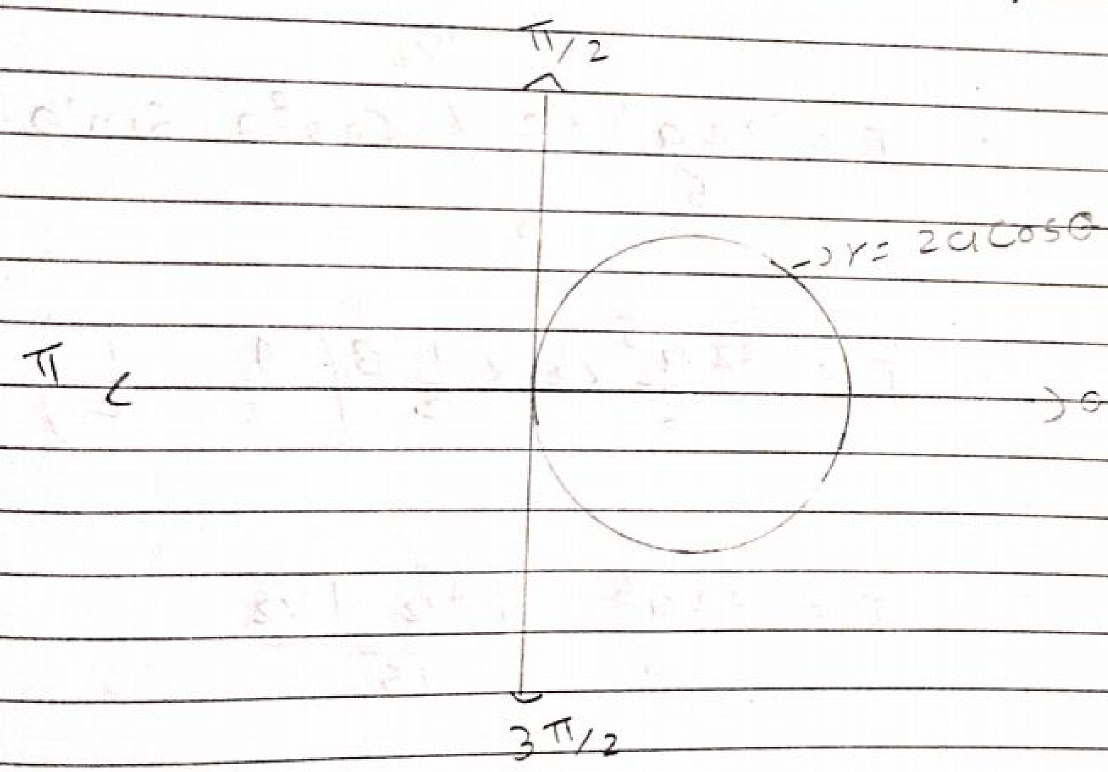
* Task - 5 : Evaluation to Double Integrals over a given Region.

.1 Evaluate $\iint_R r^4 \cos^3\theta \, dr.d\theta$ over the interior of the circle $r = 2a\cos\theta$.

Here $I = \iint_R r^4 \cos^3\theta \, dr.d\theta$ Integral bounded by Region $r = 2a\cos\theta$ Circle.



Here $I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^4 . \cos^3\theta \, dr.d\theta$

$$\therefore I = \int_{\theta=-\pi/2}^{\pi/2} \cos^3\theta \left[\frac{r^5}{5}\right]_0^{2a\cos\theta} \cdot d\theta$$

$$\therefore I = \int_{\theta=-\pi/2}^{\pi/2} \cos^3\theta \frac{(2a\cos\theta)^5}{5} \cdot d\theta$$

$$\therefore I = \frac{32a^5}{5} \int_{-\pi/2}^{\pi/2} \cos^5\theta \cdot d\theta \cos^3\theta$$

$$\therefore I = \frac{32a^5}{5} \times 2 \int_0^{\pi/2} \cos^8\theta \cdot \sin^0\theta \cdot d\theta$$

$$\therefore I = \frac{32a^5}{5} \times 2 \times \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right)$$

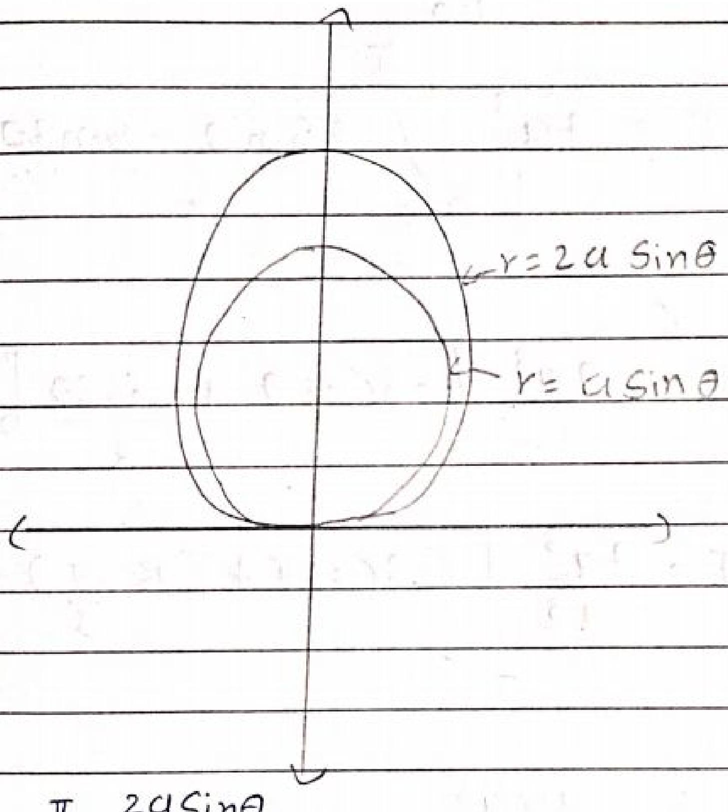
$$\therefore I = \frac{32a^5}{5} \frac{\Gamma 9/2 \cdot \Gamma 1/2}{\Gamma 5}$$

$$\therefore I = \frac{32a^5}{5} \times \frac{7/2 \times 5/2 \times 3/2 \times 1/2 \times \pi}{5 \times 4 \times 3 \times 2}$$

$$\therefore I = \frac{7\pi a^5}{4}$$

2 Evaluate $\iint_R r^2 \, dr \cdot d\theta$ over the area between $r = a\sin\theta$ and $r = 2a\sin\theta$.

Here $I = \iint_R r^2 \, dr \cdot d\theta$



$$I = \int_{\theta=0}^{\pi} \int_{r=a\sin\theta}^{2a\sin\theta} r^2 \, dr \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \left[\frac{r^3}{3}\right]_{a\sin\theta}^{2a\sin\theta} \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \frac{(2a\sin\theta)^3}{3} - \frac{(a\sin\theta)^3}{3} \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \frac{8a^3\sin^3\theta - a^3\sin^3\theta}{3} \cdot d\theta$$

$$\therefore I = \frac{7a^3}{3} \int_{\theta=0}^{\pi} \sin^3\theta \cdot d\theta$$

$$\therefore I = \frac{7a^3}{3} \int_{\theta=0}^{\pi} \frac{3\sin\theta - \sin 3\theta}{4} \cdot d\theta$$

$$\therefore I = \frac{7a^3}{12} \left[-3\cos\theta + \frac{\cos 3\theta}{3} \right]_0^{\pi}$$

$$\therefore I = \frac{7a^3}{12} \left[\left(-3\cos\pi + \cos\frac{3\pi}{3}\right) - \left(-3\cos 0 + \cos\frac{3(0)}{3}\right) \right]$$

$$\therefore I = \frac{28a^3}{9}$$

3 Evaluate $\iint_R r^2 \cdot \sin\theta \, dr \cdot d\theta$ over the cardioid $r = a(1+\cos\theta)$ above the initial line.

Here, $I = \iint_R r^2 \sin\theta \, dr \cdot d\theta$

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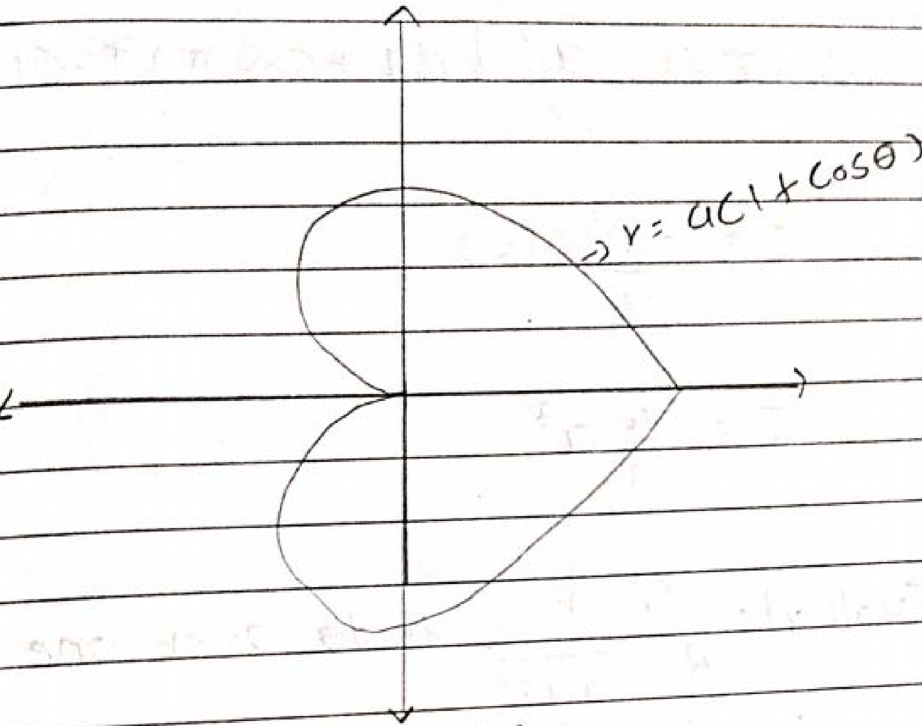


$$\therefore \quad I = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r^2 . \sin\theta \, dr . d\theta$$

$$\therefore \quad I = \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta$$

$$\therefore \quad I = \int_{\theta=0}^{\pi} \sin\theta \, \frac{(a(1+\cos\theta))^3}{3} . d\theta$$

$$\therefore \quad I = -\frac{a^3}{3} \int_{\theta=0}^{\pi} (-\sin\theta)(1+\cos\theta)^3 . d\theta$$

$$\therefore \quad I = -\frac{a^3}{3} \left[\frac{(1+\cos\theta)^4}{4} \right]_0^{\pi}$$

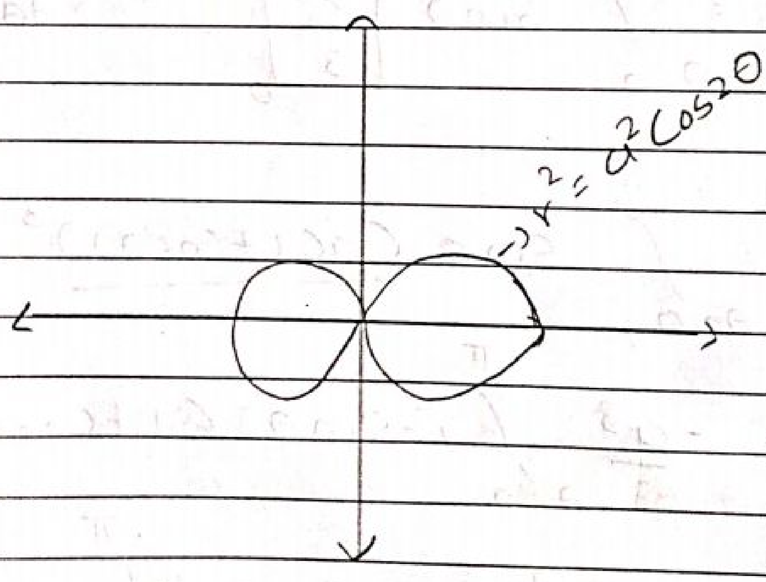
$$\therefore I = \frac{-a^3}{12}\left[(1+\cos\pi)^4 - (1+\cos 0)^4\right]$$

$$\therefore I = \frac{-a^3}{12}(-16)$$

$$\therefore I = \frac{4}{3}a^3$$

4 Evaluate $\iint_R \frac{r}{\sqrt{a^2+r^2}}\, dr.d\theta$ over one loop of Lemniscate $r^2 = a^2\cos 2\theta$

Here, $I = \iint_R \frac{r}{\sqrt{a^2+r^2}}$



$$\therefore I = \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\, dr \cdot d\theta$$

$$\therefore I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{a\sqrt{\cos 2\theta}} \frac{1}{2} \frac{2r}{\sqrt{a^2+r^2}} \, dr \cdot d\theta$$

$$\therefore I = \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \left[2(r^2+a^2)^{\frac{1}{2}} \right]_{0}^{a\sqrt{\cos 2\theta}} \cdot d\theta$$

$$\therefore I = \frac{1}{2} \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a\left((\cos 2\theta + 1)^{\frac{1}{2}} - 1\right) \cdot d\theta$$

$$\therefore I = \frac{2a}{2} \int_{-\frac{\pi}{4}}^{\pi/4} \sqrt{2\cos 2\theta} - 1 \cdot d\theta$$

$\therefore$ ~~$I = a\left[\sqrt{\ }\right.$~~

$$\therefore I = a\left[\sqrt{2}\sin\theta - \theta\right]_{-\pi/4}^{\pi/4}$$

$$\therefore I = a\left[\sqrt{2}\sin\frac{\pi}{4} - \frac{\pi}{4} - \sqrt{2}\sin\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi}{4}\right)\right]$$

$$\therefore I = a\left(2 - \frac{\pi}{2}\right)$$

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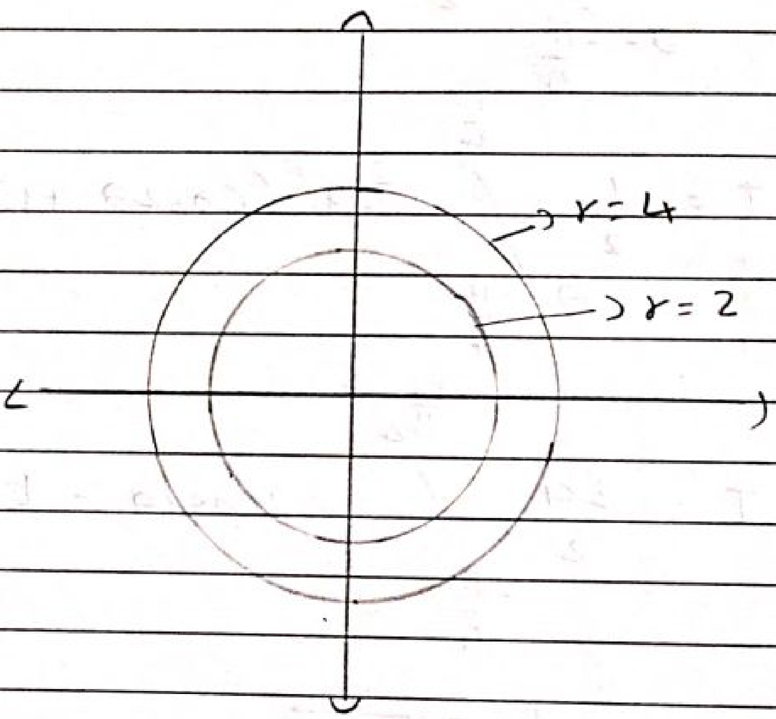
5 Evaluate $\iint_R r^3 \sin 2\theta \, dr \cdot d\theta$ over the area bounded in First quadrant R between the circles $r=2$ and $r=4$.



Here, $I = \int_{\theta=0}^{\pi/2} \int_{r=2}^{4} r^3 \sin 2\theta \, dr \cdot d\theta$

$\therefore \; I = \int_{\theta=0}^{\pi/2} \sin 2\theta \left[\frac{r^4}{4} \right]_2^4 d\theta$

$\therefore \; I = \int_{\theta=0}^{\pi/2} \sin 2\theta \left[64 - 4 \right] d\theta$

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$$\therefore I = 60 \int_{\theta=0}^{\pi/2} 2 \sin\theta \cdot \cos\theta \cdot d\theta$$

$$\therefore I = 120 \int_{0}^{\pi/2} \sin\theta \cdot \cos\theta \cdot d\theta$$

$$\therefore I = 120 \times \frac{1}{2} \beta(1, 1)$$

$$\therefore I = 60$$

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* Task : 4 Evaluation of Double Integrals in Polar Coordinates.

1 $\int_0^{\pi/2} \int_0^{1-\sin\theta} r^2 \cos\theta \, dr \cdot d\theta$

Here, $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta$

$\therefore I = \int_{\theta=0}^{\pi/2} \cos\theta \left[\frac{r^3}{3}\right]_0^{1-\sin\theta} d\theta$

$\therefore I = \int_{\theta=0}^{\pi/2} \cos\theta \frac{(1-\sin\theta)^3}{3} d\theta$

$\therefore I = -\frac{1}{3} \int_0^{\pi/2} (-\cos\theta)(1-\sin\theta)^3 \cdot d\theta$

$\therefore I = -\frac{1}{3} \left[\frac{(1-\sin\theta)^4}{4}\right]_0^{\pi/2}$

$\therefore I = -\frac{1}{12} \left[\left(1-\sin\frac{\pi}{2}\right)^4 - \left(1-\sin 0\right)^4\right]$

$$\therefore \quad I = \frac{1}{12}$$

2 $$\int_{0}^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2 \, dr \cdot d\theta$$

Here, $$I = \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a} r^2 \, dr \cdot d\theta$$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{1}{3}\left[(a)^3 - (a(1-\cos\theta))^3\right] \cdot d\theta$$

$$\therefore \quad I = \frac{a^3}{3}\int_{0}^{\pi/2} \left(1-(1-\cos\theta)\right)^3 d\theta$$

$$\therefore I = \frac{a^3}{3}\int_{0}^{\pi/2} \left[1 - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta\right] d\theta$$

$$\therefore I = \frac{a^3}{3}\int_{0}^{\pi/2} 3\cos\theta \cdot \sin^0\theta - 3\cos^2\theta \cdot \sin^0\theta + \cos^3\theta \cdot \sin^0\theta \cdot d\theta$$

$$\therefore \; I = \frac{a^3}{3}\left[\frac{3}{2}\beta\left(\frac{1}{2}, 1\right) - \frac{3}{2}\beta\left(\frac{1}{2}, \frac{3}{2}\right) + \frac{1}{2}\beta\left(\frac{1}{2}, 2\right)\right]$$

$$\therefore \; I = \frac{a^3}{3}\left[\frac{3}{2}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{2}} - \frac{3}{2}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{\Gamma 2} + \frac{1}{2}\frac{\Gamma\frac{1}{2} \times 2}{\Gamma\frac{5}{2}}\right]$$

$$\therefore \; I = \frac{a^3}{3}\left[3 - \frac{3\pi}{4} + \frac{2}{3}\right]$$

$$\therefore \; I = \frac{a^3}{3}\left[\frac{11}{3} - \frac{3\pi}{4}\right]$$

3 $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta + \phi)\, d\theta \cdot d\phi$

Here, $\displaystyle I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \sin(\theta + \phi)\, d\phi \cdot d\theta$

$$\therefore \; I = \int_{\theta=0}^{\pi/2} \left[-\cos(\theta + \phi)\right]_0^{\pi/2} \cdot d\theta$$

$$\therefore \; I = \int_{\theta=0}^{\pi/2} -\cos\left(\frac{\pi}{2} + \theta\right) - \cos\theta \cdot d\theta$$

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$$\therefore\ I = -\int_{0}^{\pi/2} -\sin\theta - \cos\theta \cdot d\theta$$

$$\therefore\ I = -\left[\cos\theta - \sin\theta\right]_{0}^{\pi/2}$$

$$\therefore\ I = -\left[\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \cos 0 + \sin 0\right]$$

$$\therefore\ I = 2$$

4 $$\int_{0}^{\pi/4}\int_{0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr \cdot d\theta$$

Here, $$I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr \cdot d\theta$$

$$\therefore\ I = \frac{1}{2}\int_{\theta=0}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{2r}{(1+r^2)^2}\, dr \cdot d\theta$$

$$\therefore\ I = \frac{1}{2}\int_{0}^{\pi/4} \left[-(1+r^2)^{-1}\right]_{0}^{\sqrt{\cos 2\theta}} \cdot d\theta$$

$$\therefore\ I = \frac{-1}{2}\int_{0}^{\pi/4} \frac{1}{1+\cos 2\theta} - 1 \cdot d\theta$$

$$\therefore I = -\frac{1}{2}\int_0^{\pi/4} \frac{1}{2}\sec^2\theta - 1 \cdot d\theta$$

$$\therefore \cancel{I = -\frac{1}{2}\int_0^{\pi}}$$

$$\therefore I = -\frac{1}{2}\left[\frac{1}{2}\tan\theta - \theta\right]_0^{\pi/4}$$

$$\therefore I = \frac{-1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$\therefore I = \frac{1}{8}(\pi - 2)$$

5 $$\int_0^a \int_0^{\cos^{-1}\left(\frac{r}{a}\right)} r\sin\theta \, dr . d\theta$$

$$\cancel{\text{Here } I = \int_{\theta=0}^{a}\int_{\theta=0}^{\cos^{-1}(r}}$$

$$\text{Here } I = \int_{r=0}^{a}\int_{\theta=0}^{\cos^{-1}\left(\frac{r}{a}\right)} r\sin\theta \, d\theta \cdot dr$$

$$\therefore I = \int_{r=0}^{a} r\left[-\cos\theta\right]_0^{\cos^{-1}\left(\frac{r}{a}\right)} . dr$$

$$\therefore I = \int_{r=0}^{a} r\left(-\cos\left(\cos^{-1}\left(\frac{r}{a}\right)\right) + \cos 0\right) \cdot dr$$

$$\therefore I = \int_{r=0}^{a} r\left(\frac{-r}{a} + 1\right) \cdot dr$$

$$\left(\because \cos(\cos^{-1}x) = x\right)$$

$$\therefore I = \int_{r=0}^{a} r - \left(\frac{r^2}{a}\right) dr$$

$$\therefore I = \left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_0^a$$

$$\therefore I = \frac{a^2}{2} - \frac{a^3}{3a}$$

$$\therefore I = \frac{a^2}{6}$$

6 $$\int_{-\pi/2}^{\pi/2} \int_{0}^{a\sin\theta} r^2 \, dr \cdot d\theta$$

Here $$I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a\sin\theta} r^2 \, dr \cdot d\theta$$

$$\therefore I = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{a\sin\theta} \cdot d\theta$$

$$\therefore I = \int_{\theta=-\pi/2}^{\pi/2} \frac{a^3}{3}\sin^3\theta \cdot d\theta$$

$$\therefore I = \frac{a^3}{3}(0)$$

(Here, $f(x) \cdot dx = 0$ Because $\sin\theta$ is odd function)

$$\therefore I = 0$$

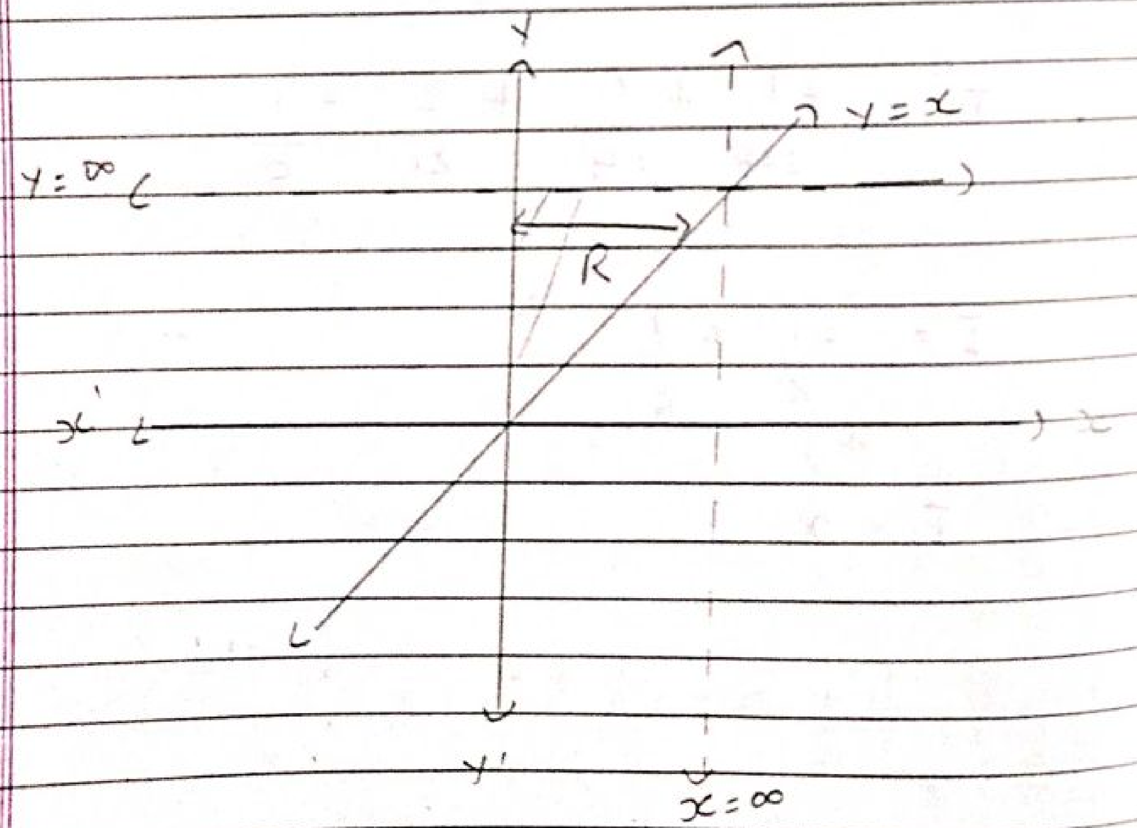
* Task : 3 Change the order of Integration

1 $\displaystyle\int_0^{\infty}\int_{x}^{\infty} \frac{e^{-y}}{y}\, dA$

Here, we want to evaluate the

$$I = \int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

We have, $x = 0$, $x = \infty$, $y = x$, $x = \infty$



$$\Rightarrow \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dx$$

$$\therefore \quad I = \int_{y=0}^{\infty} e^{-y} \, dx$$

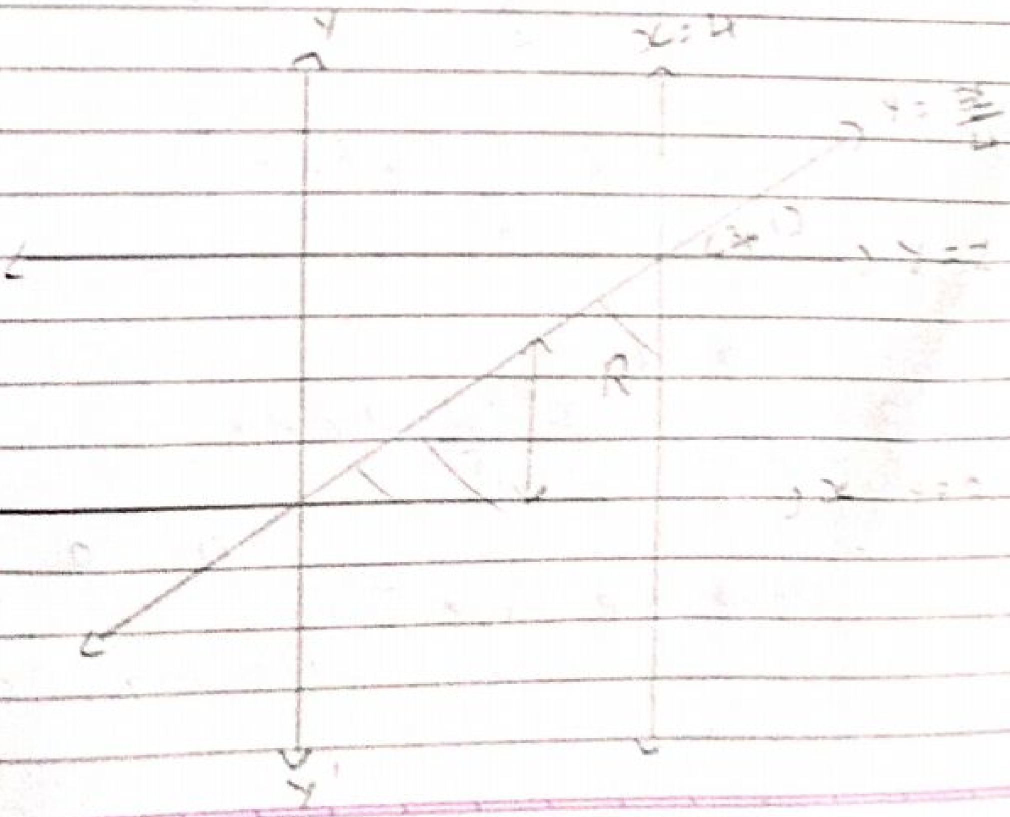
$$\therefore \quad I = \left[e^{-y}\right]_0^{\infty}$$

$$\therefore \quad I = 1$$

2 $\displaystyle\int_0^1 \int_{4y}^{4} e^{x^2} \, dx \, dy$

Here $\displaystyle I = \int_{y=0}^{1} \int_{x=4y}^{4} e^{x^2} \, dx \, dy$

We have, $y = 0$, $y = 1$, $x = 4$ and $x = 4y$



$$\Rightarrow \int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy = \int_{x=0}^4 \int_{y=0}^{x/4} e^{x^2}\, dy\, dx$$

$$I = \int_{x=0}^4 e^{x^2} \cdot [y]_0^{x/4}\, dx$$

$$\therefore I = \int_{x=0}^4 \frac{x}{4} \cdot e^{x^2}\, dx$$

$$\therefore I = \frac{1}{8} \int_{x=0}^4 e^{x^2} \cdot 2x \cdot dx$$

$$\therefore I = \frac{1}{8} [e^{x^2}]_0^4$$

$$\therefore I = \frac{1}{8} [e^{16} - e^0]$$

$$\therefore I = \frac{1}{8} [e^{16} - 1]$$

3 $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$

Here we have $I = \int_{y=0}^a \int_{x=y}^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$

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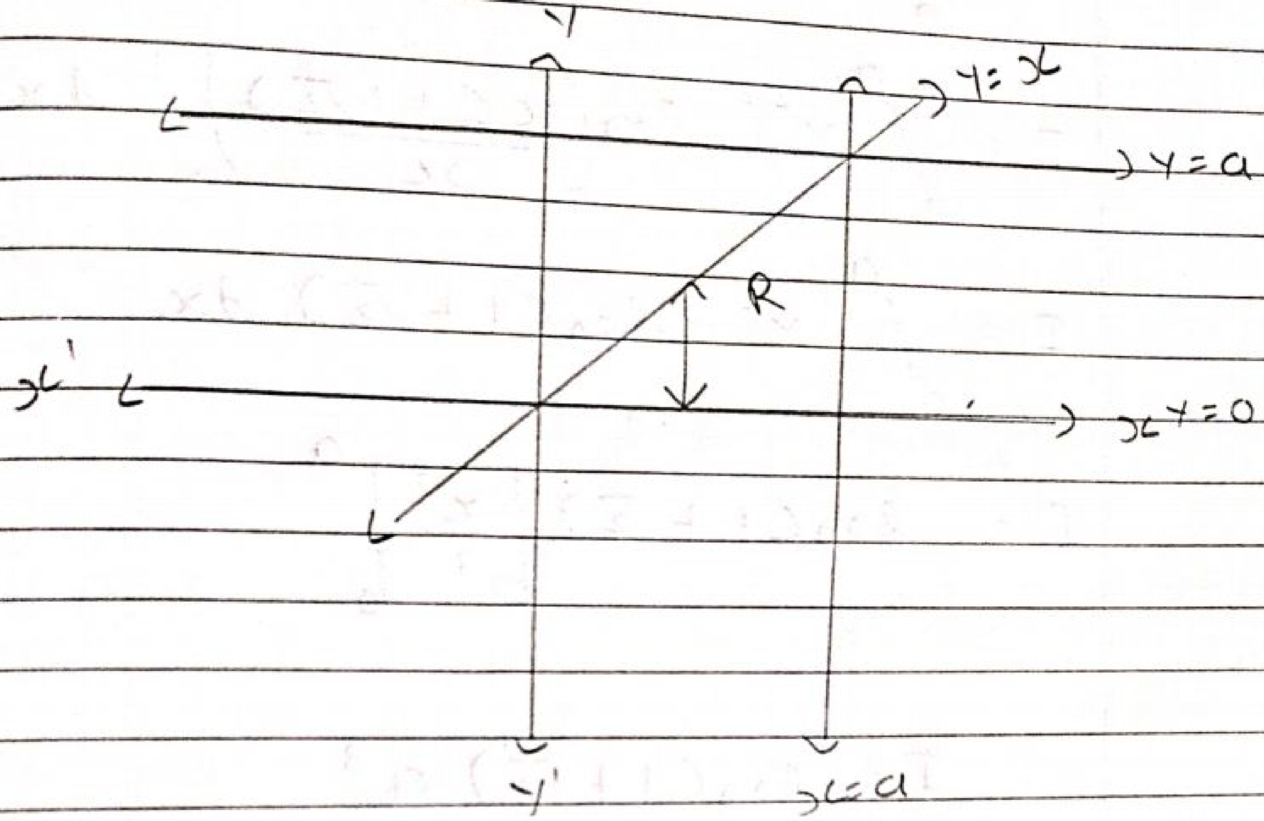
We have $y=0$, $y=a$, $y=x$, $x=a$



$$\Rightarrow \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{x=0}^a \int_{y=0}^x \frac{x^2}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$\therefore I = \int_{x=0}^a \int_{y=0}^x x^2 \cdot \cancel{lo} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$\therefore I = \int_{x=0}^a \left[x^2 \log\left(y + \sqrt{x^2+y^2}\right) \right]_0^x dx$$

$$\therefore I = \int_0^a x^2 \left[\log\left(x + \sqrt{x^2+x^2}\right) - \log\left(\sqrt{x^2}\right) \right] dx$$

$$I = \int_0^a x^2 \left[\log(x + x\sqrt{2}) - \log x \right] dx$$

$$I = \int_0^a x^2 \left(\log\left(\frac{x(1+\sqrt{2})}{x} \right) \right) dx$$

$$I = \int_0^a x^2 \cdot \log(1+\sqrt{2})\, dx$$

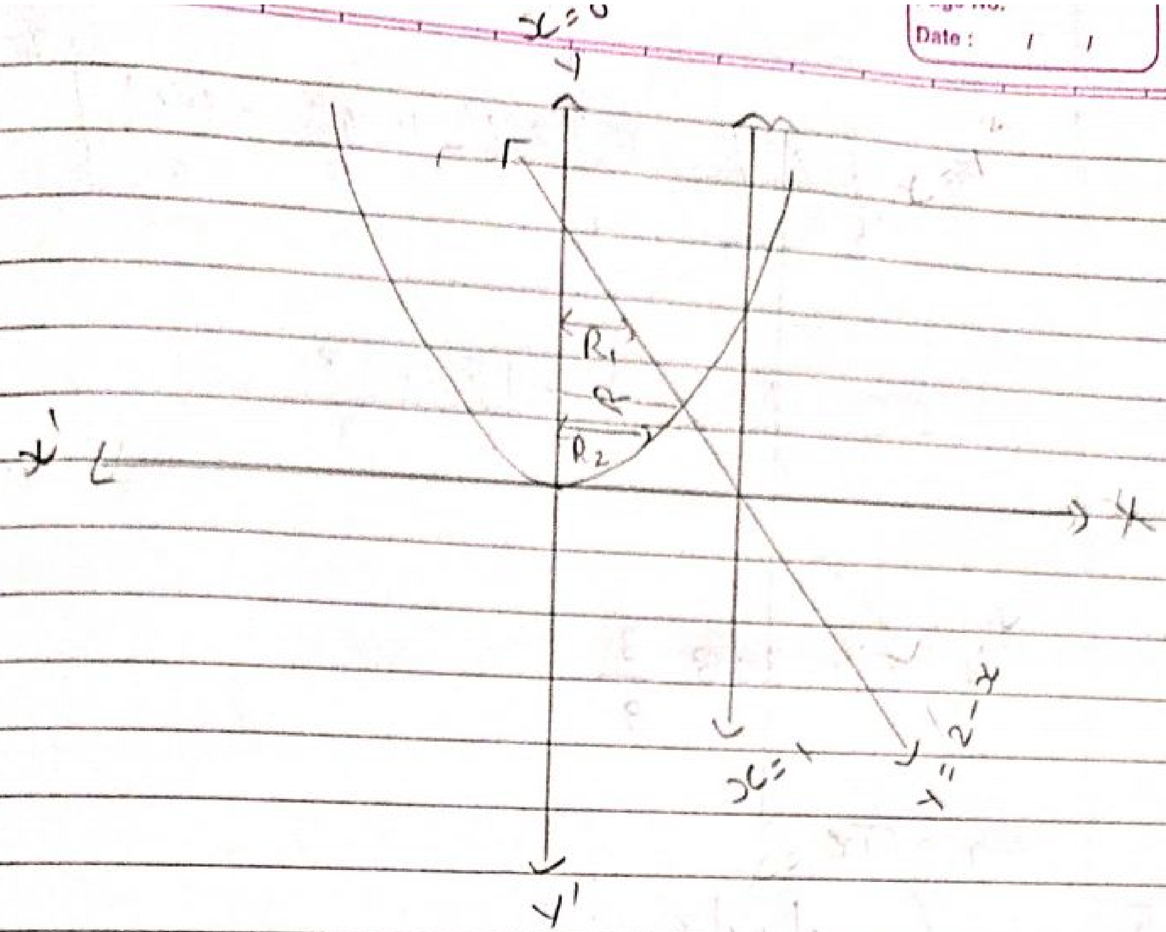
$$I = \log(1+\sqrt{2}) \left[\frac{x^3}{3} \right]_0^a$$

$$\therefore \quad I = \log(1+\sqrt{2}) \frac{a^3}{3}$$

4 $\displaystyle\int_0^1 \int_{x^2}^{2-x} xy \, dA$

Here $I = \displaystyle\int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \cdot dx$

We have $x = 0$, $x = 1$, $y = x^2$ and $y = 2 - x$



$$\Rightarrow \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\cdot dx = \int_{y=0}$$

$$I = \iint_{R_1} xy\,dy\,dx + \iint_{R_2} xy\,dy\,dx$$

$$\therefore\ I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$\therefore\ I = \int_{0}^{1} y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_{1}^{2} y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$\therefore\ I = \int_{0}^{1} \frac{y^2}{2}\cdot dy + \int_{1}^{2} y\,\frac{(4-4y+y^2)}{2}\,dy$$

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$$\therefore I = \left[\frac{y^3}{6}\right]_0^1 + \left[\frac{2y^2}{2} + \frac{y^4}{8} - \frac{4y^3}{6}\right]_1^2$$

$$\therefore I = \frac{1}{6} + 3 - \frac{15}{8} + \frac{18}{3}$$

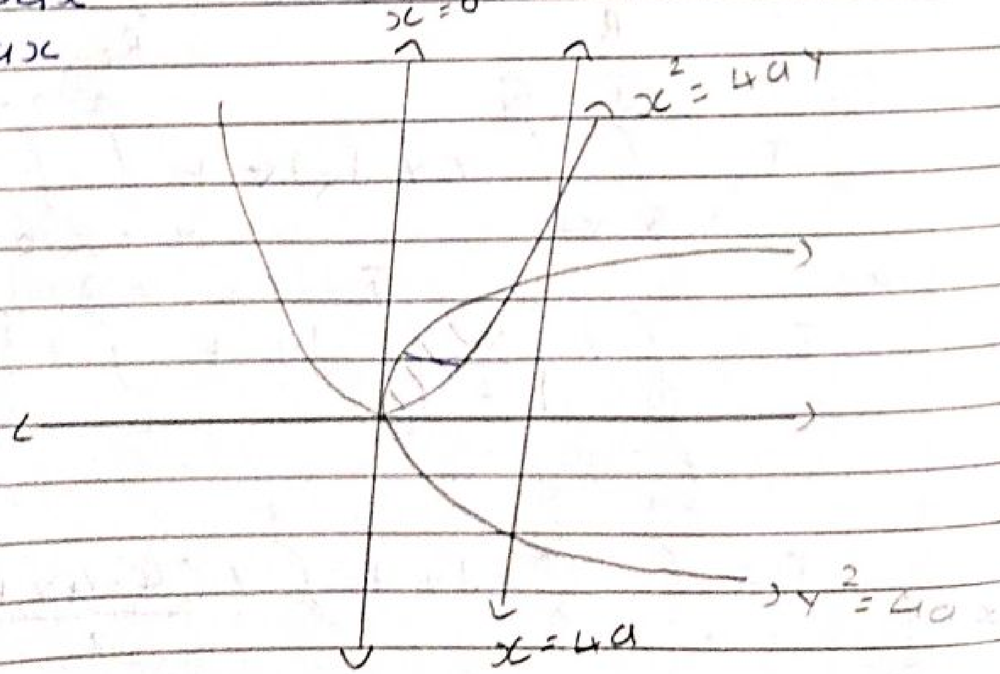
$$\therefore I = \cancel{318}\ \frac{3}{8}$$

5 $$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

Here, $$I = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} 1\ dy\,dx$$

$y = 2\sqrt{ax}$

$y^2 = 4ax$



$$\Rightarrow \int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} 1 \, dy\, dx = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} 1 \, dx\, dy$$

$$I = \int_{y=0}^{4a} \Big[x \Big]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$\therefore \quad I = \int_{y=0}^{4a} 2\sqrt{ay} - \frac{y^2}{4a} \cdot dy$$

$$\therefore \quad I = \int_{0}^{4a} 2\sqrt{a}\, y^{1/2} - \frac{1}{4a} y^2 \cdot dy$$

$$\therefore \quad I = \left[2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3} \right]_0^{4a}$$

$$\therefore \quad I = \frac{4}{3}\sqrt{a} \cdot (a)^{3/2} - \frac{1}{4a}\frac{(4a)^3}{3}$$

$$\therefore \quad I = \frac{4}{3} a \cdot a^3 - \frac{1}{4a}\frac{(64a^3)}{3}$$

$$\therefore \quad I = \frac{32a^2 - 16a^2}{3}$$

$$\therefore \quad I = \frac{16a^2}{3}$$

6 $$\int_0^{1/2}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$$

Here $$I = \int_{y=0}^{1/2}\int_{x=0}^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$$

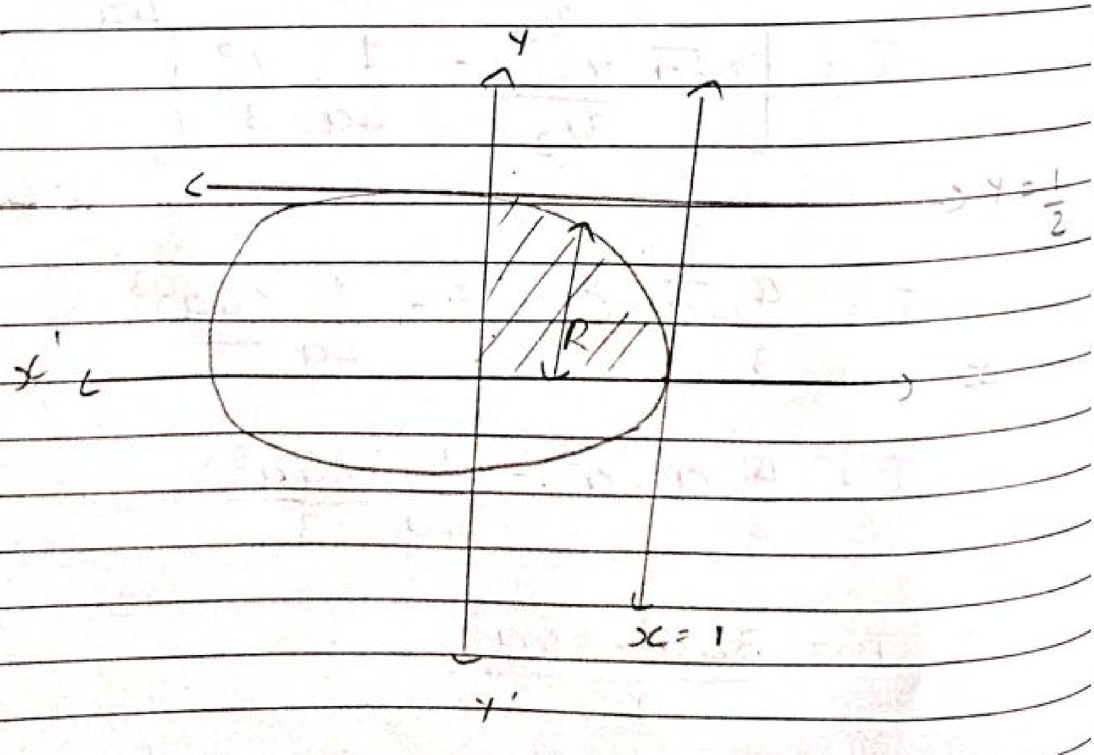
We have $y=0,\ y=1/2$

$x=0,\ x=\sqrt{1-4y^2}$

$\therefore x^2 = 1-4y^2$

$\therefore x^2+4y^2 = 1$

$\therefore \dfrac{x^2}{(1)^2} + \dfrac{y^2}{(1/2)^2} = 1$



$$I = \int_{x=0}^{1} \int_{y=0}^{\frac{1}{2}\sqrt{1-x^2}} \frac{1+x^2}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$I = \int_{x=0}^{1} \frac{1+x^2}{\sqrt{1-x^2}} \int_{y=0}^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$I = \int_{x=0}^{1} \frac{1+x^2}{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}} \right) \right]_0^{\frac{1}{2}\sqrt{1-x^2}} \cdot dx$$

$$I = \int_{x=0}^{1} \frac{1+x^2}{\sqrt{1-x^2}} \left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) \right) dx$$

$$I = \int_0^1 \frac{\pi}{6} \cdot \frac{2(1-x^2)}{\sqrt{1-x^2}}\, dx$$

$$I = \frac{\pi}{6} \int_0^1 \frac{2}{\sqrt{1-x^2}} - \frac{(1-x^2)}{\sqrt{1-x^2}}\, dx$$

$$I = \frac{\pi}{6} \left[2\sin^{-1}(x) - \frac{1}{2}\sin^{-1}(1) \right]_0^1$$

$$= \frac{\pi}{6} \left(\frac{3}{2} \sin^{-1}(1) \right)$$

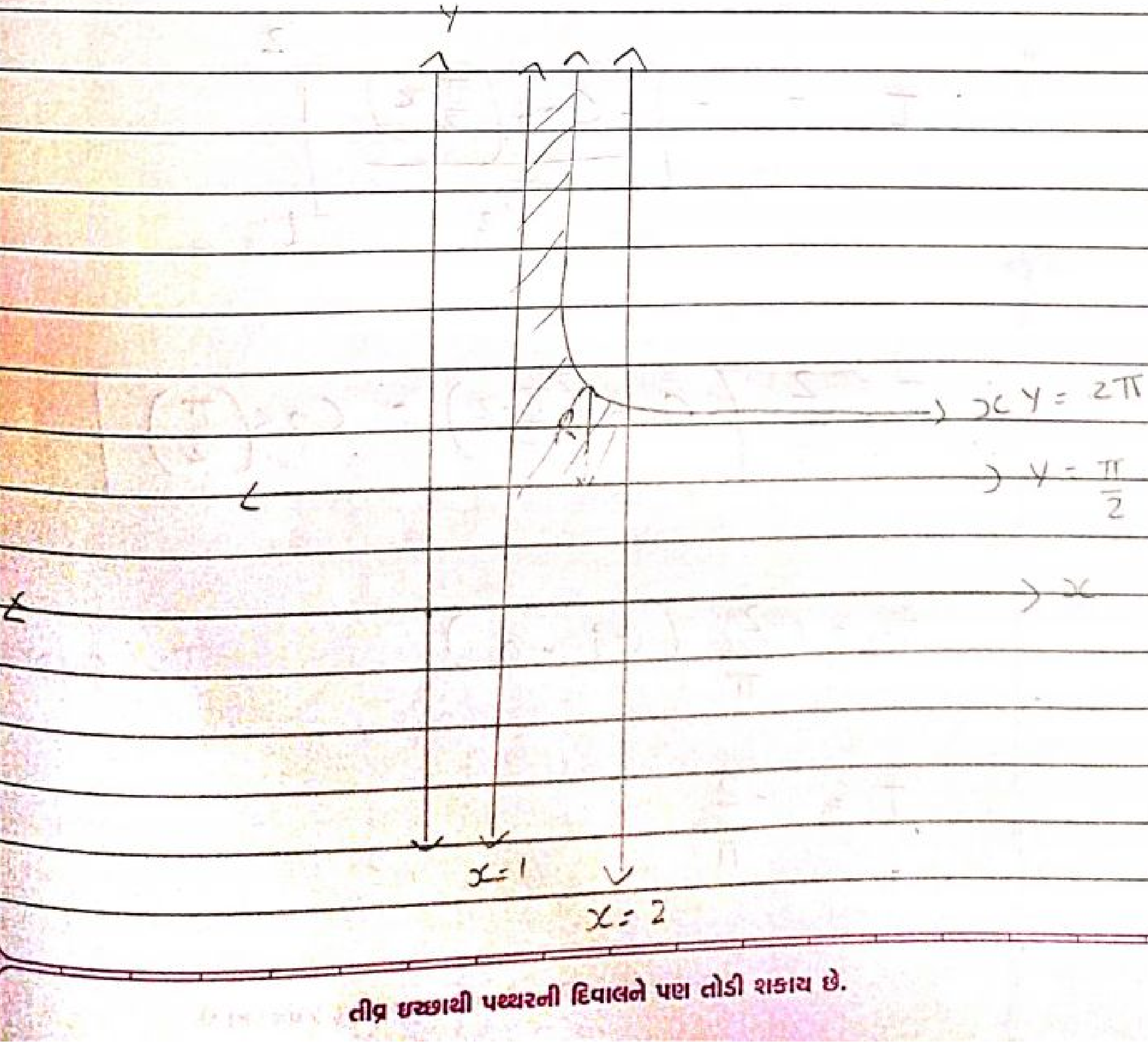
$$= \frac{\pi^2}{8}$$

* Task : 2 Calculate the Integral over a Given region.

1 Evaluate $\iint_R x \cos(xy)\, dA$, where R is the Region in closed by $x = 1$, $x = 2$, $y = \pi/2$ and $x = \frac{2\pi}{y}$.

Here, we want to evaluate $\iint_R x \cos(xy)\, dA$ where R is region,

$x = 1$, $x = 2$, $y = \pi/2$ and $y = \frac{2\pi}{x}$



Here: $I = \int_{x=1}^{2} \int_{y=\pi/2}^{2\pi/x} x \cos(xy)\, dy\, dx$

$\therefore I = \int_{x=1}^{2} x \left[\frac{\sin(xy)}{x} \right]_{\frac{\pi}{2}}^{\frac{2\pi}{x}} dx$

$\therefore I = \int_{x=1}^{2} \sin\left(x \cdot \frac{2\pi}{x}\right) - \sin\left(x \cdot \frac{\pi}{2}\right) dx$

$\therefore I = \int_{x=1}^{2} \sin(2\pi) - \sin\left(\frac{\pi}{2} x\right) dx$

$\therefore I = -\left[\frac{-\cos\left(\frac{\pi}{2} x\right)}{\pi/2} \right]_1^2$

$\therefore I = \frac{2}{\pi} \left[\cos\left(\frac{\pi}{2} \cdot 2\right) - \cos\left(\frac{\pi}{2}\right) \right]$

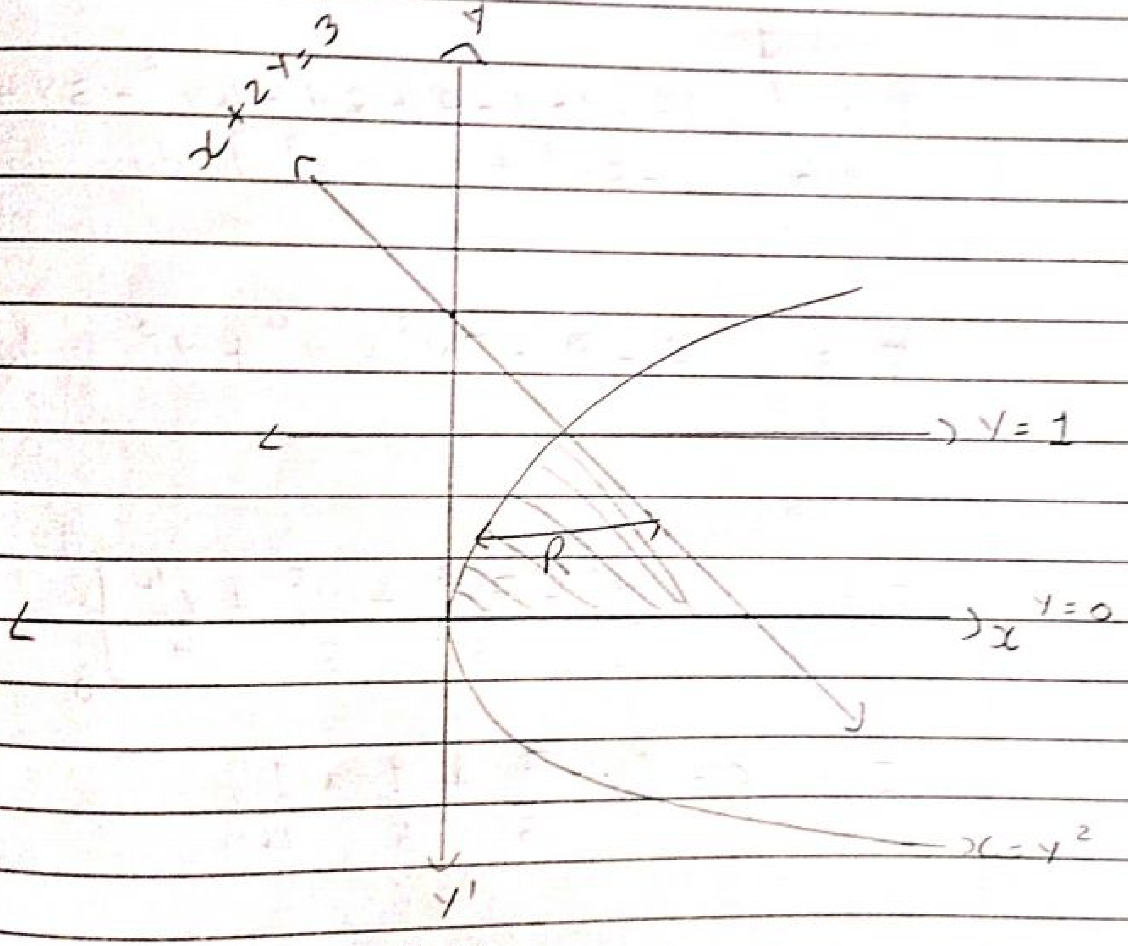
$\therefore I = \frac{2}{\pi} \left(-1 - 0 \right)$

$\therefore I = \frac{-2}{\pi}$

2 Evaluate $\iint_R (5-2x-y)\,dy\,dx$, where R is the region bounded by $y=0$, $x+2y=3$, $x=y^2$.

Here, Region bounded by,

$y=0$, $x+2y=3$, $x=y^2$



Here $I = \int_{y=0}^{1} \int_{x=y^2}^{3-2y} (5-2x-y)\,dx\,dy$

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$$\therefore I = \int_{y=0}^{1} \left[5x - \frac{2x^2}{2} - yx \right]_{y^2}^{3-2y} \cdot dy$$

$$\therefore I = \int_{y=0}^{1} \Big[5(3-2y) - (3-2y)^2 - y(3-2y) \Big] - \left[5y^2 - y^4 - y^3 \right] \cdot dy$$

$$\therefore I = \int_{y=0}^{1} \big[15 - 10y - 9 + 12y - 4y^2 - 3y + 2y^2 - 5y^2 + y^4 + y^3 \big] \, dy$$

$$\therefore I = \int_{y=0}^{1} 6 - y - 7y^2 + y^4 + y^3 \cdot dy$$

$$\therefore I = \left[6y - \frac{y^2}{2} - \frac{7y^3}{3} + \frac{y^5}{5} + \frac{y^4}{4} \right]_0^1$$

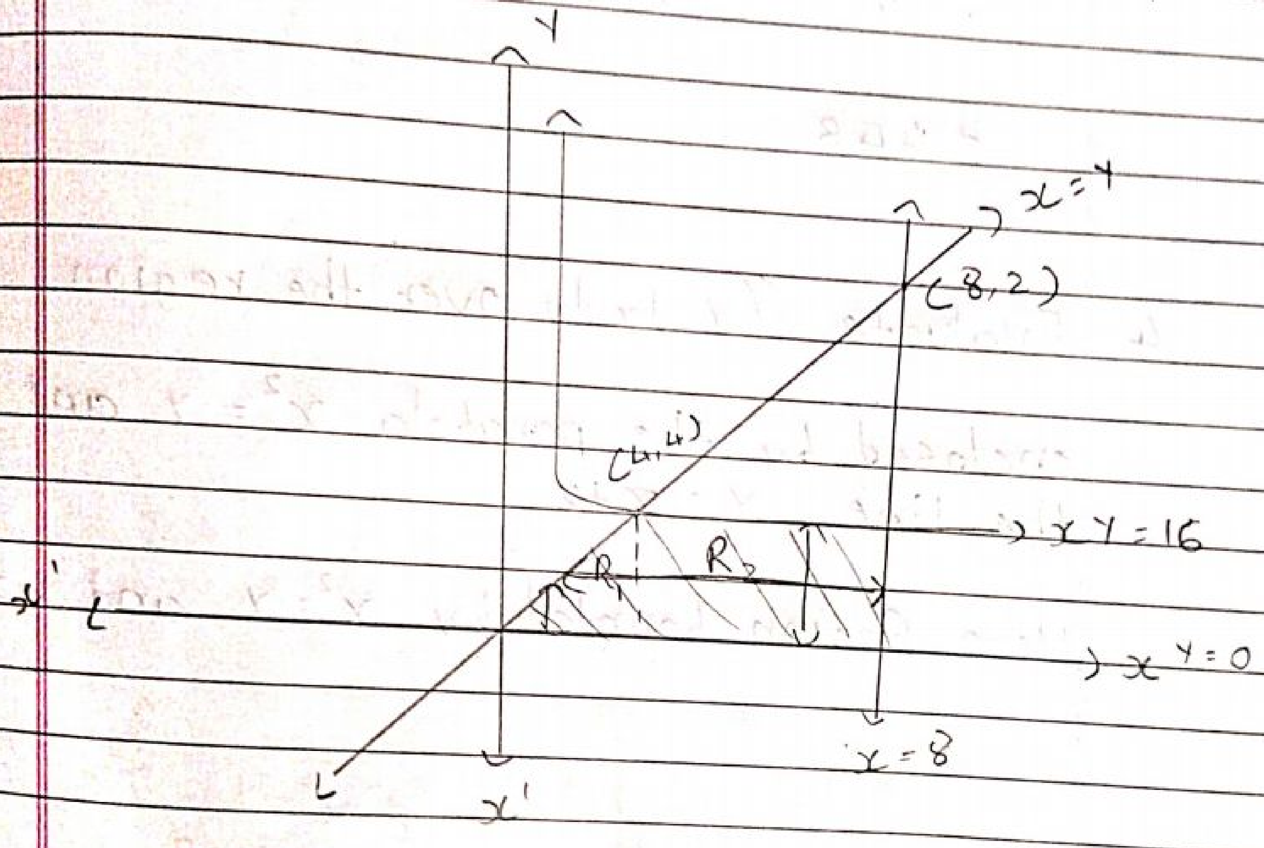
$$\therefore I = 6 - \frac{1}{2} - \frac{7}{3} + \frac{1}{5} + \frac{1}{4}$$

$$\therefore I = \frac{217}{60}$$

3 Evaluate $\iint_R x^2 \cdot dA$. where R is the region bounded by $xy = 16$ lines $y = x$, $y = 0$, $x = 8$ in First quadrant hyperbola.

Here Region bounded by,

$xy = 16$, $y = x$, $y = 0$ and $x = 8$ lines.



Here, $I = \iint_{R_1} x^2 \cdot dA + \iint_{R_2} x^2 \cdot dA$

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 \, dy \cdot dx + \int_{x=4}^{8} \int_{y=0}^{16/x} x^2 \cdot dy \cdot dx$$

$$= \int_{x=0}^{4} x^2 \left[y \right]_0^{x} \cdot dx + \int_{x=4}^{8} x^2 \left[y \right]_0^{16/x} \cdot dx$$

$$I = \int_0^4 x^3 \, dx + \int_4^8 \frac{16}{x} \cdot x^2 \cdot dx$$

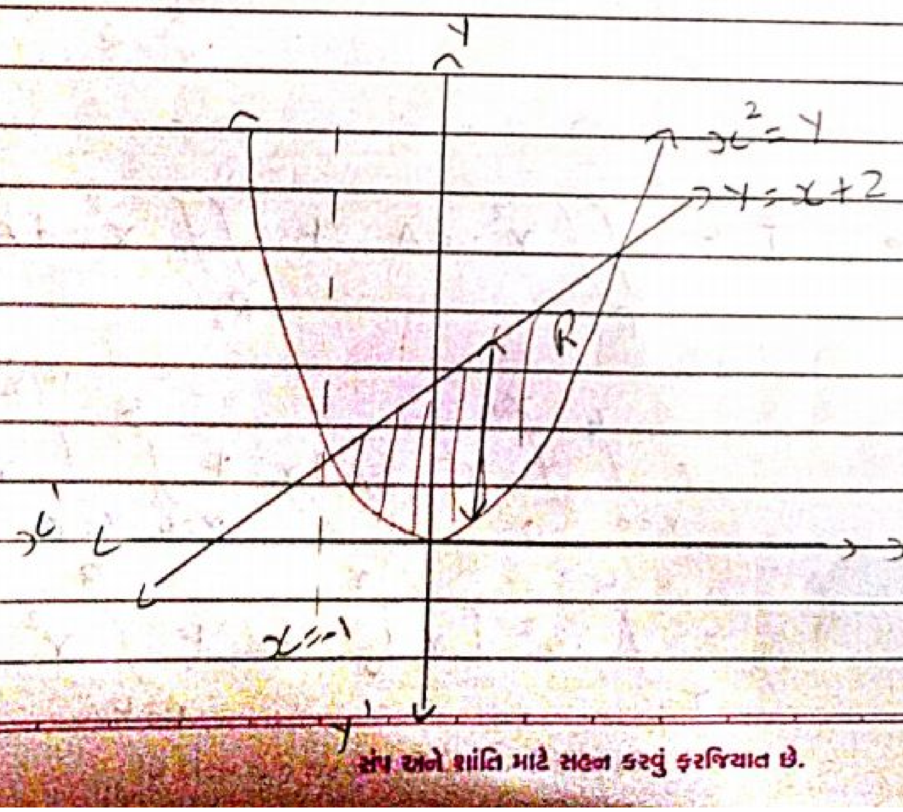
$$I = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8$$

$$I = \frac{(4)^4}{4} + 16\left(\frac{64}{2} - \frac{16}{2}\right)$$

$$= 448$$

4 Evaluate $\iint y \, dx\,dy$ over the region enclosed by the parabola $x^2 = y$ and the line $y = x+2$

Here, Region bounded by $x^2 = y$ and $y = x+2$



Here $I = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} y \, dy \, dx$

$\therefore I = \int_{x=-1}^{2} \left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx$

$\therefore I = \int_{x=-1}^{2} \frac{1}{2}\left((x+2)^2 - x^4\right) dx$

$\therefore I = \int_{x=-1}^{2} \frac{1}{2}\left(x^2+4x+4-x^4\right) dx$

$\therefore I = \frac{1}{2}\int_{x=-1}^{2} x^2 - x^4 + 4x + 4 \cdot dx$

$\therefore I = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{4x^2}{2} + 4x\right]_{-1}^{2}$

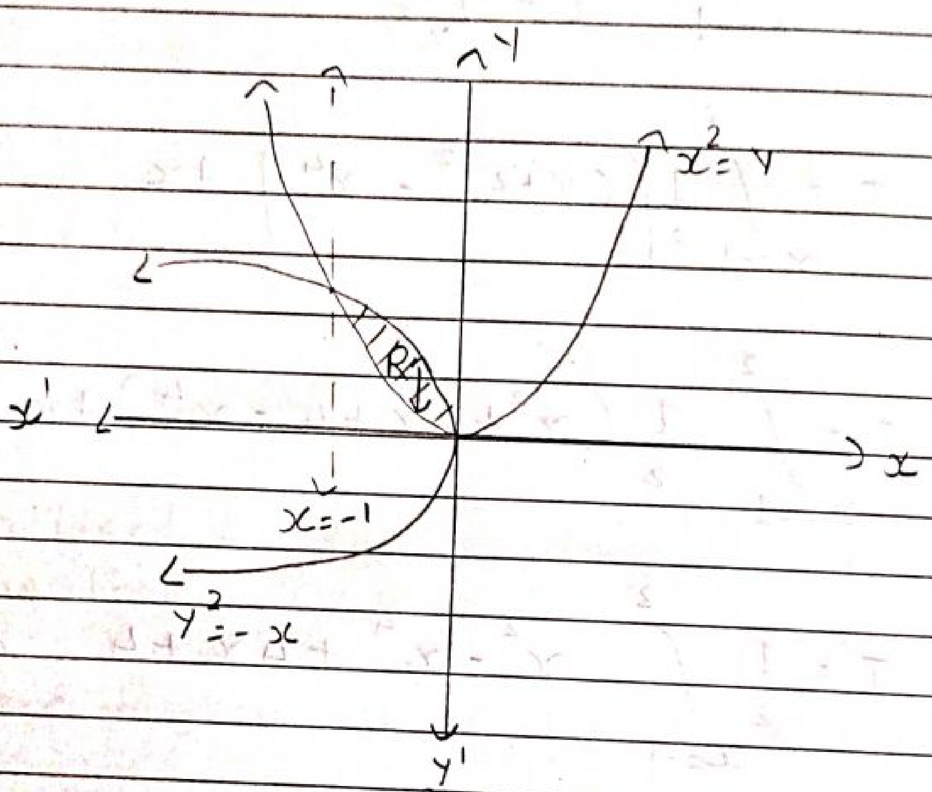
$\therefore I = \frac{1}{2}\left(\frac{8}{3} - \frac{32}{5} + 2(4) + 8\right) - \left(\frac{-1}{3} + \frac{1}{5} + 2 - 4\right)$

$\therefore I = \frac{1}{2}\left(\frac{8}{3} - \frac{32}{5} + 8 + 8\right) + \frac{1}{3} - \frac{1}{5} + 2$

$\therefore I = 36/5$

5 Evaluate $\iint xy(x+y)\,dx\,dy$ over the region enclosed by the parabolas $x^2 = y$ and $y^2 = -x$

Here, Region bonded by $x^2 = y$ and $y^2 = -x$



Here, $I = \int_{x=-1}^{0} \int_{y=-\sqrt{x}}^{x^2} x^2 y + x y^2 \, dy\, dx$

$\therefore\ I = \int_{x=-1}^{0} x^2 \left[\frac{y^2}{2}\right]_{-\sqrt{x}}^{x^2} + x \left[\frac{y^3}{3}\right]_{-\sqrt{x}}^{x^2} dx$

$\therefore\ I = \int_{x=-1}^{0} x^2 \left[\frac{x^4}{2} - \frac{x}{2}\right] + x \left[\frac{-x^6}{3} - \frac{(-\sqrt{x})^3}{3}\right] dx$

$$\therefore I = \int_{x=-1}^{0} \frac{x^6}{2} - \frac{x^3}{2} - \frac{x^7}{3} + \frac{x^{5/2}}{3} \cdot dx$$

$$\cancel{\therefore I = \int_{-1}^{0} x}$$

$$\cancel{\therefore I = \left[\frac{x^8}{} \right.}$$

$$\therefore I = \left[\frac{x^7}{14} - \frac{x^4}{8} - \frac{x^8}{24} + \frac{2x^{7/2}}{21} \right]_{-1}^{0}$$

$$\therefore I = \frac{-1}{24} + \frac{1}{14} + \frac{2}{21} - \frac{1}{8}$$

$$\therefore I = \frac{-1}{6} + \frac{1}{6}$$

$$\therefore I = 0$$

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Unit : 6 Multiple Integrals and its Application

* Task 1: Evaluate the Double Integrals.

2 $\int_0^1 \int_0^x e^{y/x} \cdot dy \cdot dx$

Here, $I = \int_0^1 \int_0^x e^{y/x} dy \cdot dx$ given.

$\therefore I = \int_{x=0}^{1} \int_{y=0}^{x} e^{y/x} dy \cdot dx$

$= \int_{x=0}^{1} \left[\frac{e^{y/x}}{1/x} \right]_0^x \cdot dx$

$= \int_{x=0}^{1} x \left(e^{x/x} - e^0 \right) \cdot dx$

$= \int_{x=0}^{1} x \left(e^1 - 1 \right) \cdot dx$

$$I = \left[\frac{x^2}{2}\right]_0^1 (e^1 - 1)$$

$$I = \frac{1}{2}(e^1 - 1)$$

3 $$\int_0^1 \int_0^y x y e^{-x^2} \, dA$$

$$\text{Here } I = \int_0^1 \int_0^y x e^{-x^2} y \cdot dA$$

$$\therefore I = \int_{y=0}^{1} \int_{x=0}^{y} x e^{-x^2} \cdot y \cdot dx\, dy$$

$$I = -\frac{1}{2} \int_{y=0}^{1} \int_{x=0}^{y} (-2x) e^{-x^2} \cdot y \cdot dx\, dy$$

$$= -\frac{1}{2} \int_{y=0}^{1} \left[e^{-x^2}\right]_0^y \cdot y \cdot dy$$

$$= -\frac{1}{2} \int_{y=0}^{1} e^{-y^2} \cdot y \cdot dy$$

$$= \frac{-1(-)}{4} \int_{y=0}^{1} (-2y) \cdot e^{-y^2} \cdot dy$$

$$I = \frac{1}{4}\left[e^{-y^2}\right]_0^1$$

$$I = \frac{1}{4}e^{-1}$$

$$I = \frac{1}{464e}$$

5 $$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$$

Here, $$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dy\, dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2} \cdot dy\, dx$$

$$I = \int_{x=0}^{1} \cancel{\int_{y=0}^{\sqrt{1+x^2}}} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} \cdot dx$$

$$I = \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} \cdot dx$$

$$I = \frac{\pi}{4} \int_{x=0}^{1} \frac{1}{\sqrt{(1)^2 + (x^2)}} \cdot dx$$

$$I = \frac{\pi}{4} \left[\log \left(1 + \sqrt{1+x^2}\right) \right]_0^1$$

$$I = \frac{\pi}{4} \log\left(1 + \sqrt{2}\right)$$

1 $$\int_0^a \int_0^b \frac{1}{xy} \, dx \cdot dy$$

Here $$I = \int_0^a \int_0^b \frac{1}{xy} \, dx \cdot dy$$

$$I = \int_{y=0}^{a} \int_{x=0}^{b} \frac{1}{xy} \cdot dx \cdot dy$$

$$I = \int_{y=0}^{a} \left[\log x \right]_0^b \cdot \frac{1}{y} \cdot dy$$

$$I = \int_0^a \log b \cdot \frac{1}{y} \cdot dy$$

$$I = \log b \left[\log y \right]_0^a$$

$$I = \log a \cdot \log b$$

4 $$\int_0^a \int_{x/a}^{x} \frac{x}{x^2+y^2} \, dy \cdot dx$$

Here $$I = \int_0^a \int_{x/a}^{x} \frac{x}{x^2+y^2} \, dy \cdot dx$$

$$I = \int_{x=0}^{a} \int_{y=x/a}^{x} \frac{x}{x^2+y^2} \, dy \, dx$$

$$I = \int_{x=0}^{a} x \cdot \frac{1}{x} \left[\tan^{-1}\left(\frac{y}{x}\right) \right]_{x/a}^{x} dx$$

$$I = \int_0^a x \cdot \frac{1}{x} \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{a}\right) \right] dx$$

$$I = \int_0^a \frac{\pi}{4} \frac{x}{x} - \tan^{-1}\left(\frac{1}{a}\right) \frac{x}{x} \cdot dx$$

$$I = \int_0^a \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{a}\right) dx$$

$$= \left[\frac{\pi}{4}x - \tan^{-1}\left(\frac{1}{a}\right)x\right]_0^a$$

$$I = \frac{\pi}{4}a - \tan^{-1}\left(\frac{1}{a}\right)a$$

6 $$\int_0^a \int_0^{\sqrt{ax-x^2}} x^2 + y^2 \; dx \cdot dy$$

Here $$I = \int_0^a \int_0^{\sqrt{ax-x^2}} x^2 + y^2 \; dx\, dy$$

$$\therefore I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{ax-x^2}} x^2 + y^2 \; dy\, dx$$

$$\therefore I = \int_0^a \left[x^2 y + \left[\frac{y^3}{3}\right]\right]_0^{\sqrt{ax-x^2}} dx$$

$$\therefore I = \int_0^a x^2\sqrt{ax-x^2} + \frac{\left(\sqrt{ax-x^2}\right)^3}{3} \; dx$$

$$\therefore I = \int_0^a x^2\sqrt{ax-x^2} + \frac{\sqrt{ax-x^2}\,(ax-x^2)}{3} \; dx$$

$$I = \int_0^a x^2\sqrt{ax-x^2} + \frac{ax\sqrt{ax-x^2}}{3} - \frac{x^2\sqrt{ax-x^2}}{3}\cdot dx$$

$$I = \int_0^a \frac{2}{3}x^2\sqrt{ax-x^2} + \frac{ax}{3}\sqrt{ax-x^2}\cdot dx$$

$$I = \int_{x=0}^a x^2(ax-x^2)^{\frac{1}{2}} + \frac{(ax-x^2)^{3/2}}{3}\cdot dx$$

$$I = \int_0^a x^{5/2}(a-x)^{1/2}\cdot dx + \frac{1}{3}\int_0^a x^{3/2}(a-x)^{3/2}\cdot dx$$

Here, we take $x = at$

$\therefore dx = a\cdot dt$

x	a	1
t	0	0

$$\therefore I = \int_0^1 a^{5/2}t^{5/2}(a-at)^{1/2} + \frac{1}{3}\int_0^1 a^{3/2}t^{3/2}(a-at)^{3/2}$$

$$\therefore I = \int_0^1 a^4 t^{5/2}(1-t)^{1/2}\cdot dt + \frac{1}{3}\int_0^1 a^4 t^{3/2}(1-t)^{3/2}\,dt$$

$$\therefore I = a^4 \int_0^1 t^{\frac{7}{2}-1} (1-t)^{\frac{3}{2}-1} \cdot dt$$

$$+ \frac{a^4}{3} \int_0^1 t^{\frac{5}{2}-1} (1-t)^{\frac{5}{2}-1} \cdot dt$$

$$\therefore I = a^4 \frac{\Gamma 7/2 \cdot \Gamma 3/2}{\Gamma 6} + \frac{a^4}{3} \frac{\Gamma 5/2 \cdot \Gamma 5/2}{\Gamma 5}$$

$$\therefore I = \frac{a^4 \, 6 \pi^2}{4 \times 4 \times 4 \times 2}$$

$$\therefore I = \frac{3 a^4 \pi^2}{64}$$

* Task 6: Change of variables from Cartesian to Polar Coordinates.

1 Evaluate $\iint_R \frac{4xy}{x^2+y^2} e^{-x^2-y^2}\, dx.dy$ over the region bounded by the circle,

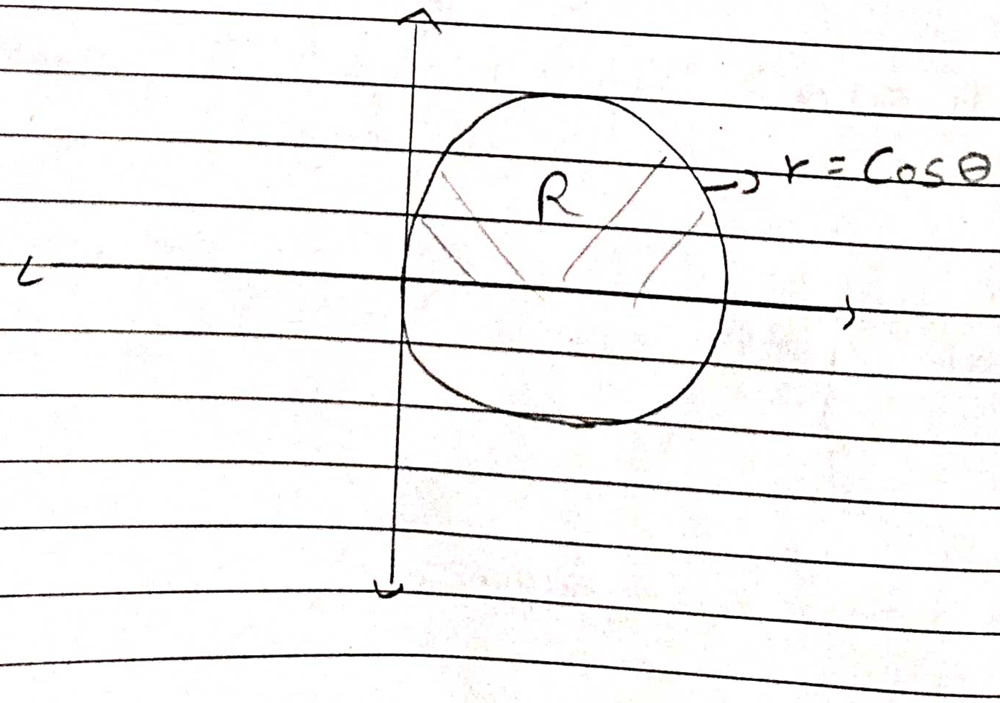
$$x^2+y^2-x=0 \text{ in first quadrant}$$

Here $x = r\cos\theta$ and $y = r\sin\theta$

Now $x^2+y^2-x=0 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - r\cos\theta =$

$$\Rightarrow r^2 = r\cos\theta$$

$$\Rightarrow r = \cos\theta$$



$$\Rightarrow \iint \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} \frac{4r\cos\theta\, r\sin\theta}{r^2} e^{-r^2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} 2r\cdot\sin 2\theta\cdot e^{-r^2}\, dr\cdot d\theta$$

$$= -\int_{0}^{\pi/2}\int_{r=0}^{\cos\theta} (-2r)\sin 2\theta\, e^{-r^2}\, dr\cdot d\theta$$

$$= -\int_0^{\pi/2} \sin 2\theta \left[e^{-r^2}\right]_0^{\cos\theta} \cdot d\theta$$

$$= -\int_0^{\pi/2} \sin 2\theta \left[e^{-\cos^2\theta} - e^0\right] d\theta$$

$$= -\left[e^{-\cos^2\theta} + \frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{e}$$

2 Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$

Here we take $x = r\cos\theta$, $y = r\sin\theta$

$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r \cdot dr \cdot d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \int_0^\infty (-2r) \cdot e^{-r^2} \, dr \cdot d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} \left[-e^{-r^2}\right]_0^\infty \cdot d\theta$$

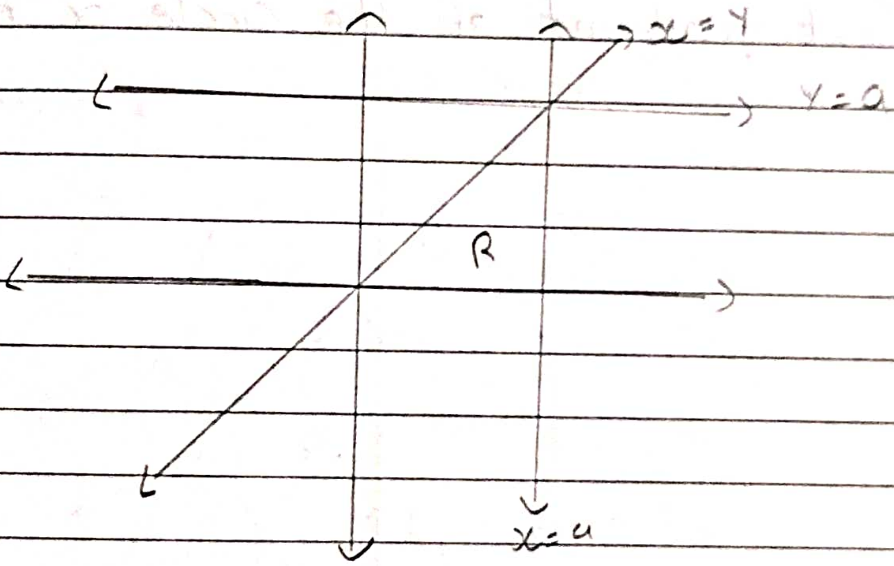
$$= -\frac{1}{2} \int_0^{\pi/2} (-1) \cdot d\theta$$

$$= \frac{1}{2} \left[\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

3 Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ by transforming into polar coordinates.

$$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx \cdot dy$$



$$I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{a/\cos\theta} \frac{r\cos\theta}{r^2} \cdot r \cdot dr \cdot d\theta$$

$$I = \int_{\theta=0}^{\pi/4} \left[1 \right]_0^{a/\cos\theta} \cos\theta \, d\theta$$

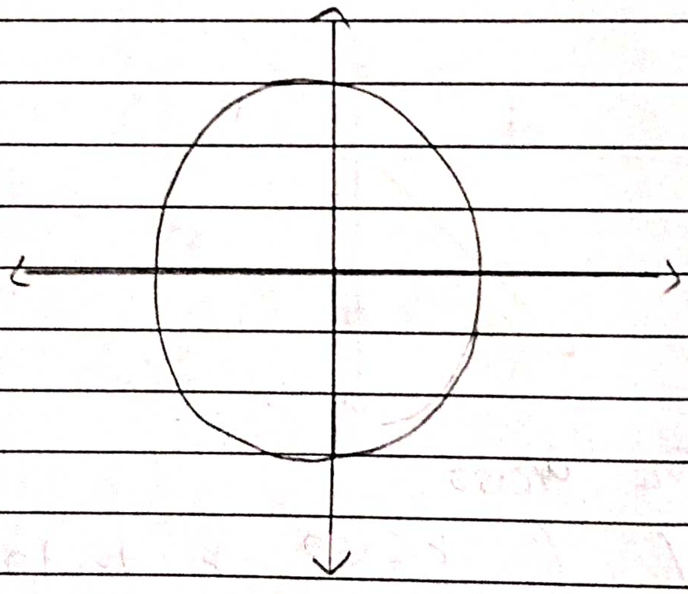
$$I = \int_{\theta=0}^{\pi/4} \frac{a}{\cos\theta} \cdot \cos\theta \cdot d\theta$$

$$I = \left[\theta\right]_0^{\pi/4} \cdot a$$

$$I = \frac{\pi}{4} a$$

4 Evaluate $\iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$ over the

First quadrant of the circle $x^2+y^2=1$



Here, $x = r\cos\theta$ and $y = r\sin\theta$

Hance, $x^2+y^2=1$

$\therefore r^2\cos^2\theta + r^2\sin^2\theta = 1$

$\therefore r = 1$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, r\cdot dr\cdot d\theta$$

Let $r^2 = \cos 2t$

$\therefore 2r\cdot dr = -2\sin 2t\cdot dt$

$\therefore r\cdot dr = -\sin 2t\cdot dt$

$r = 0 \Rightarrow t = \pi/4$

$r = 1 \Rightarrow t = 0$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{t=\pi/4}^{0} \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}\, (-\sin 2t)\, dt\cdot d\theta$$

$$\therefore I = \int_{0}^{\pi/2} \int_{\pi/4}^{0} \sqrt{\frac{2\sin^2 t}{2\cos^2 t}}\, (-\sin 2t)\, dt\cdot d\theta$$

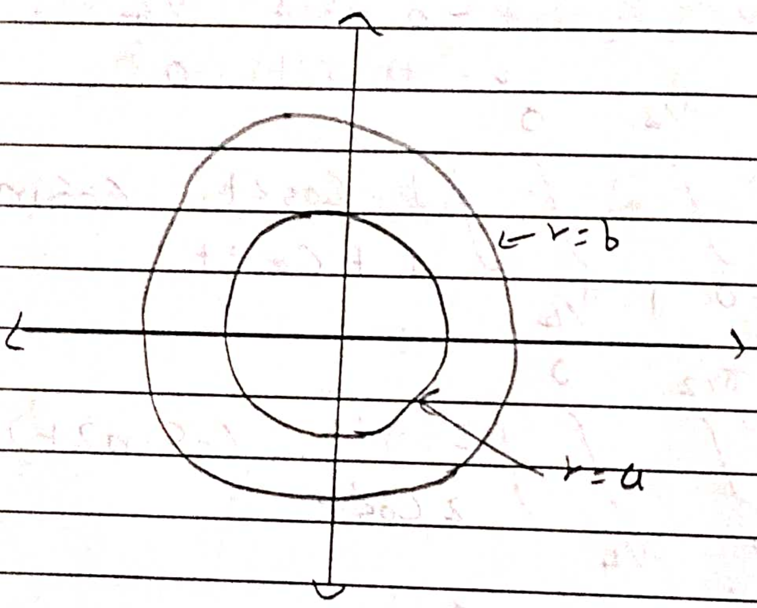
$$\therefore I = +\int_{0}^{\pi/2} \int_{\pi/4}^{0} (1-\cos 2t)\, dt\cdot d\theta$$

$$\therefore I = \int_{0}^{\pi/2} \left(t - \frac{\sin 2t}{2}\right)_{0}^{\pi/4} d\theta$$

$$\therefore I = \int_{0}^{\pi/2} \frac{\pi}{4} - \frac{1}{2}\, d\theta$$

$$\therefore I = \frac{\pi}{8}(\pi - 2)$$

5 Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}$ over the region bounded by the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$



$$I = \int_{\theta=0}^{2\pi}\int_{r=b}^{a} \frac{r^2\cos^2 a\, r^2\sin^2 a}{r^2(\cos^2 a+\sin^2 a)} \cdot r\, dr \cdot d\theta$$

$$I = \int_{\theta=0}^{2\pi}\int_{b}^{a} r^3 \cos^2 a \sin^2 a \, dr\, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \cos^2\theta \sin^2\theta \left[\frac{r^4}{4}\right]_b^a d\theta$$

$$I = \frac{1}{4}\int_0^{2\pi} 4\cos^2\theta \sin^2\theta \left(\frac{a^4 - b^4}{4}\right) d\theta$$

$$I = \int_0^{2\pi} \frac{\sin^2 2\theta}{4} \cdot \frac{a^4 - b^4}{4}\, d\theta$$

$$I = \frac{a^4 - b^4}{\cancel{4}\cancel{8}\cancel{4}8}\left[\frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$I = \frac{a^4 - b^4}{\cancel{6}8}\left[\frac{2\pi}{4}\right]$$

$$I = \frac{2\pi}{32}(a^4 - b^4) = \frac{\pi}{16}(a^4 - b^4)$$

* Task : 7 : Change of Variables from Cartesion to other Coordinates.

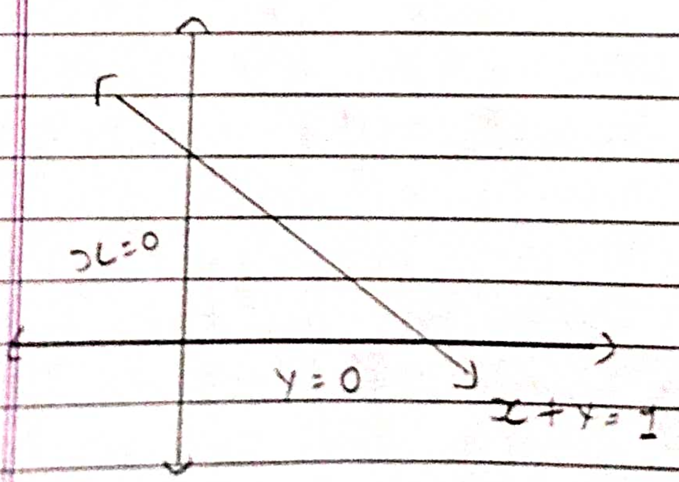
1 Using the transformation $x - y = u$, $x + y = v$, evaluate $\iint_R \cos\frac{x-y}{x+y}\,dx\,dy$

over the region bounded by the line $x = 0$, $y = 0$, $x + y = 1$

$\Rightarrow$ $x - y = u$, $x + y = v$

$\therefore x = \frac{u+v}{2}$, $y = \frac{v-u}{2}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

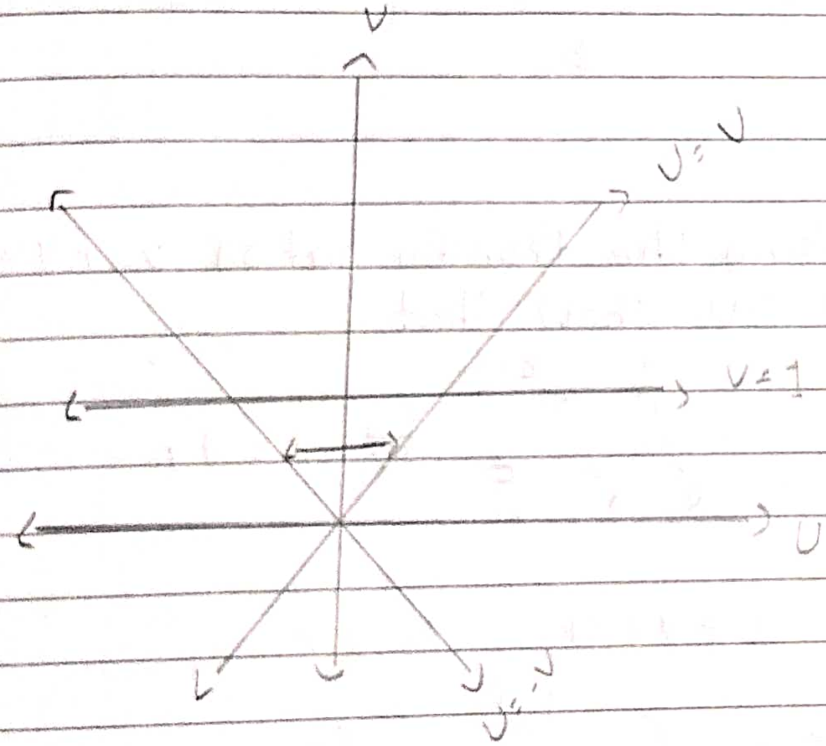


⇒ Given region,

$x = 0 \rightarrow 0 = \frac{u+v}{2} \Rightarrow u = -v$

$y = 0 \rightarrow 0 = \frac{u-v}{2} \Rightarrow u = v$

$x + y = 1 \rightarrow \frac{u+v}{2} + \frac{u-v}{2} = v = 1$



$$I = \int_{v=0}^{1} \int_{u=-v}^{v} \cos\left(\frac{u}{v}\right) \frac{1}{2} \, du \cdot dv$$

$$I = \frac{1}{2} \int_{v=0}^{1} \left[\frac{\sin(u/v)}{1/v} \right]_{-v}^{v} \cdot dv$$

$$\therefore \quad I = \frac{1}{2} \int_{v=0}^{1} v\,[\sin(1) - \sin(-1)]\, dv$$

$$= \frac{1}{2} \sin(1) \int_{0}^{1} 2v \, dv$$

$$= \frac{1}{2} \sin(1) \int \left[\frac{2v^2}{2} \right]_0^1 d$$

$$= \frac{\sin(1)}{2}$$

2 Using the transformation $x + y = u$ and $y = uv$ show that.

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \, dx \, dy$$

$\Rightarrow \quad x + y = u, \quad y = uv$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} e^{\frac{y}{x+y}} \, dy \, dx$$

$\rightarrow \quad x + uv = u$

$$\therefore \quad x = u(1-v)$$
$$y = uv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

$\Rightarrow x = 0 \to u(1-v) = 0 \to u = u\overset{0}{\cancel{v}} \to = v = 0$

$\Rightarrow x = 1 \to u(1-v) = 1 \to u = uv \to v = \cancel{0}1$

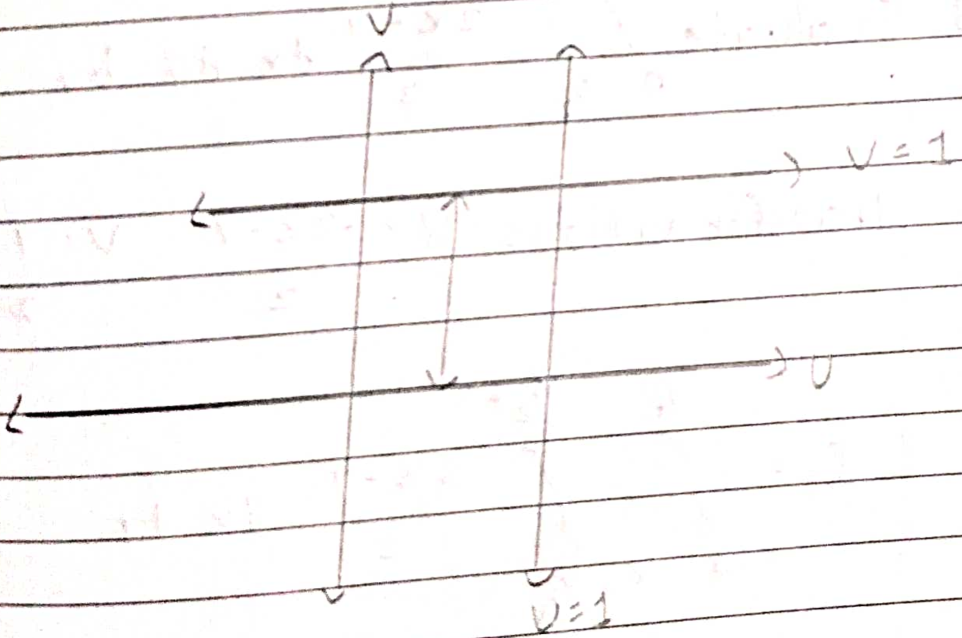
$\Rightarrow y = 0 \to 0 = uv \Rightarrow u = 0 , v = 0$

$\Rightarrow y = 1-x \to x+y = 1 \to u + uv + uv = 1$

$\to u + 2uv = 1$

$\to u(1+2v) = 1$

$\to u = 1 , v = 0$



$$I = \int_{v=0}^{1} \int_{u=0}^{1} e^{v} \cdot u \cdot du \cdot dv$$

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$$I = \int_{v=0}^{1} e^{v} \cdot \left[\frac{u^2}{2}\right]_0^1 dv$$

$$I = \int_{v=0}^{1} \frac{e^{v}}{2} dv$$

$$I = \frac{1}{2}\left[e^{v}\right]_0^1$$

$$I = \frac{1}{2}\left[e^{1} - e^{0}\right]$$

$$I = \frac{1}{2}\left[e - 1\right]$$

3 Evaluate $\int_{0}^{4}\int_{y/2}^{\frac{y}{2}+1} \frac{2x-y}{2} \, dx.dy$ by transformations $u = \frac{-2x-y}{2}$, $v = \frac{y}{2}$

$$I = \int_{y=0}^{4}\int_{x=y/2}^{y/2+1} \frac{2x-y}{2} \, dx \, dy$$

$\Rightarrow$ $2u = -2x - y$

$\therefore$ $2u = -2x - 2v$

$\Rightarrow U = x - \frac{y}{2}$

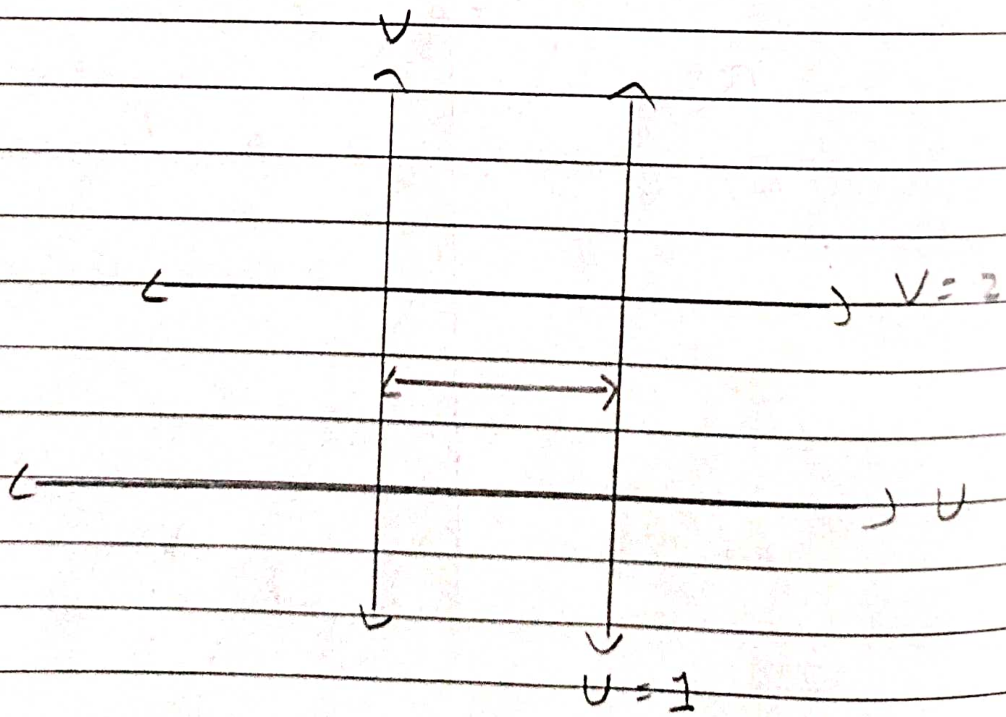
$\rightarrow x = \frac{y}{2} \rightarrow U = \frac{y}{2} - \frac{y}{2} = 0$

$\rightarrow x = \frac{y}{2} + 1 \rightarrow U = \frac{y}{2} + 1 - \frac{y}{2} = 1$

$\Rightarrow V = \frac{y}{2}$

$\rightarrow y \Rightarrow 0 \rightarrow V = 0$

$\rightarrow y \Rightarrow 4 \rightarrow V = 2$



$$I = \int_{v=0}^{2} \int_{u=0}^{1} U \cdot J \, du \, dv$$

$$\Rightarrow J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}$$

$$\Rightarrow I = \int_{v=0}^{2} \frac{1}{2}\left[\frac{u^2}{2}\right]_0^1 dv$$

$$I = \int_0^2 \frac{\cancel{1}}{\cancel{2}} \frac{1}{4} dv$$

$$I = \left[\frac{v}{4}\right]_0^2$$

$$I = \frac{1}{2}$$

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* Task 8 : Evaluation of triple integrals when limits given, Over the given region and Change of variable

1 Evaluate $\int_{1}^{3}\int_{1}^{1/x}\int_{0}^{\sqrt{xy}} xy\, dz\, dy\, dx$

$$I = \int_{x=1}^{3}\int_{y=1}^{1/x}\int_{z=0}^{\sqrt{xy}} xy\, dz\, dy\, dx$$

$$I = \int_{x=1}^{3}\int_{y=1}^{1/x} \left[z\right]_{0}^{\sqrt{xy}} xy\, dy\, dx$$

$$I = \int_{x=1}^{3}\int_{y=1}^{1/x} \sqrt{xy}\cdot xy\, dy\, dx$$

$$I = \int_{x=1}^{3}\int_{y=1}^{1/x} \cancel{x^{3/2}}\; x\sqrt{x}\cdot y\cdot\sqrt{y}\, dy\, dx$$

$$I = \int_{x=1}^{3} x\sqrt{x}\left[\frac{2}{3} y^{3/2}\right]_{1}^{1/x} dx$$

$$I = \int_{x=1}^{3} x\sqrt{x}\cdot\frac{2}{3}\left[\frac{1}{x^{3/2}} - 1\right] dx$$

$$I = \frac{2}{3}\left(\log x - \frac{2}{3}x^{3/2}\right)_1^3$$

$$I = \frac{2}{3}\left[\log 3 - 2\sqrt{3} + \frac{2}{3}\right]$$

2 Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2}(1-x^2-y^2)\, dy\, dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2} - \frac{x^3y}{2} - \frac{y^3x}{2}\; dy \cdot dx$$

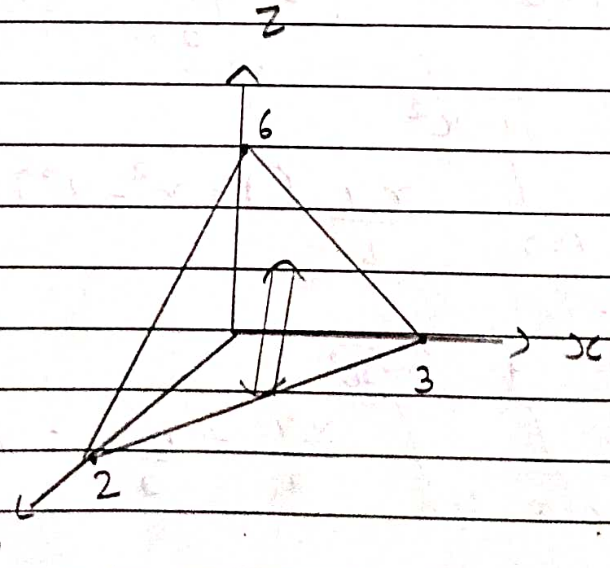
$$I = \frac{1}{2}\int_{x=0}^{1}\left[\frac{x^2}{2}\frac{y^2}{2} - \frac{x^4y^4}{4} - \frac{y^4x^2}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$\therefore I = \frac{1}{8}\int_0^1 x(1-2x^2+x^4)\,dx$$

$$\therefore I = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$\therefore I = \frac{1}{48}$$

3 Evaluate $\iiint_V 2x\,dv$, where V is the region under the plane $2x+3y+z=6$ that lines in the first octant



$$I = \int_{x=0}^{3}\int_{y=0}^{\frac{6-2x}{3}}\int_{z=0}^{6-2x-3y} 2x\;dz\cdot dy\cdot dx$$

$$I = 2\int_{x=0}^{3}\int_{y=0}^{\frac{6-2x}{3}} x\left[z\right]_0^{6-2x-3y} dy\cdot dx$$

$$I = 2\int_{x=0}^{3}\int_{y=0}^{\frac{6-2x}{3}} (6-2x-3y)x\,dz\cdot dx$$

$$I = 2\int_{x=0}^{3} x\left(6-2xy-3\left[\frac{y^2}{2}\right]_0^{\frac{6-2x}{3}}\right)dx$$

$$I = 2\int_{x=0}^{3} x\left(6-2x\cdot\left(\frac{6-2x}{3}\right)-3\frac{(6-2x)^2}{6}\right.dx$$

$$I = \frac{4}{3}\int_{x=0}^{3} 9x+x^3-6x\cdot dx$$

$$I = \frac{4}{3}\left[\frac{3x^2}{2}+\frac{x^4}{4}\right]_0^3$$

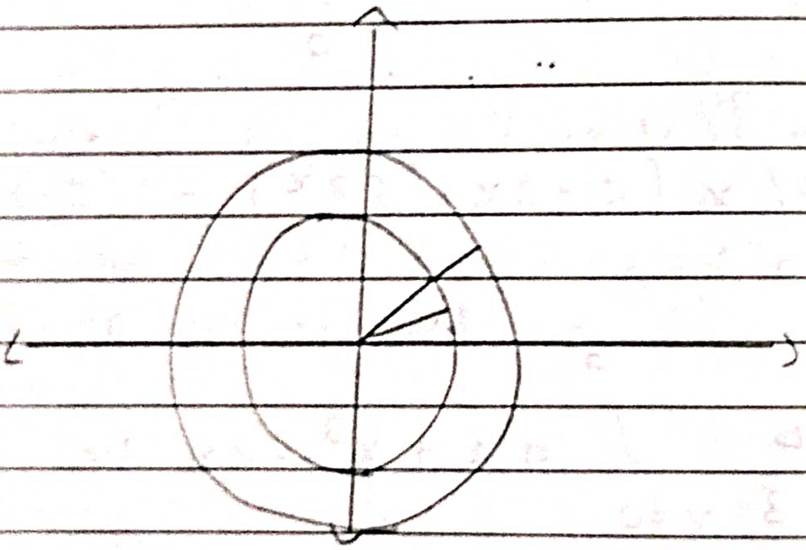
$$I = 9$$

4 Evaluate $\iiint_V \frac{1}{(x^2+y^2+z^2)^{1/2}}\, dx\, dy$ over the region bounded by the spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, $a > b > 0$

Here, $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



$$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{b} \frac{r^2\sin\theta}{r}\, dr\, d\theta\, d\phi$$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_a^b \sin\theta\, d\theta\, d\phi$$

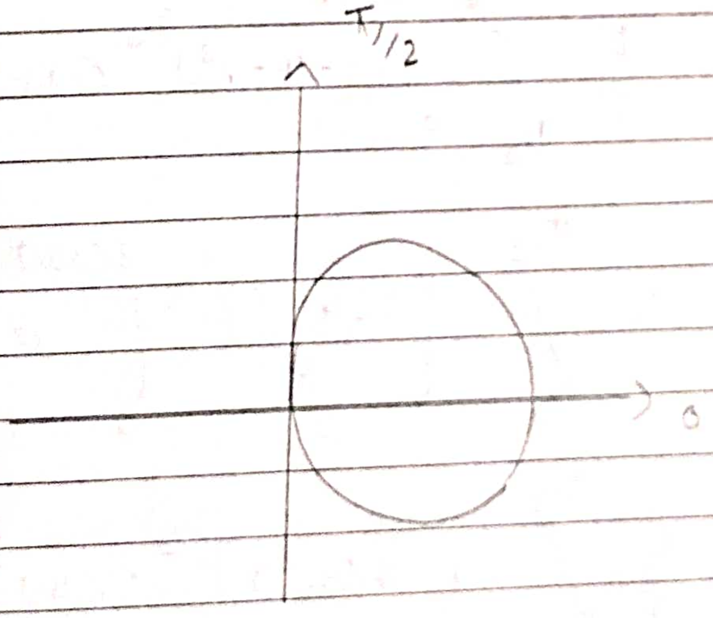
$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{[a^2-b^2]}{2} \sin\theta\, d\theta\, d\phi$$

$$I = \frac{a^2 - b^2}{2} \int_{\phi=0}^{2\pi} [\cos\theta]_0^{\pi} \, d\phi$$

$$I = \frac{a^2 - b^2}{2} \int_0^{2\pi} (\cos\pi - \cos 0) \, d\phi$$

$$I = 4\pi \frac{(a^2 - b^2)}{2}$$

5 Evaluate $\iiint z^2 \, dx \, dy \, dz$ over the region $x^2 + y^2 + z^2 = 4$ by and the cylinder $x^2 + y^2 = 2x$



Here, $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$

$\Rightarrow$ $x^2 + y^2 + z^2 = 4$

$r^2 + z^2 = 4$

$z = \pm\sqrt{4 - r^2}$

$\Rightarrow \quad x^2 + y^2 = 2x$

$\therefore \quad r^2 = 2r\cos\theta$

$\therefore \quad r = 2\cos\theta$

$\rightarrow \quad \theta = -\pi/2 + \pi/2$

$$\therefore I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z \cdot r \, dr \, d\theta \, d\phi$$

$$I = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \frac{-2}{3} (4-r^2)^{3/2} \, r \, dr \, d\theta$$

$$I = \frac{-1}{3} \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} [-4-r^2)^{3/2} (-2r) \, dr \, d\theta$$

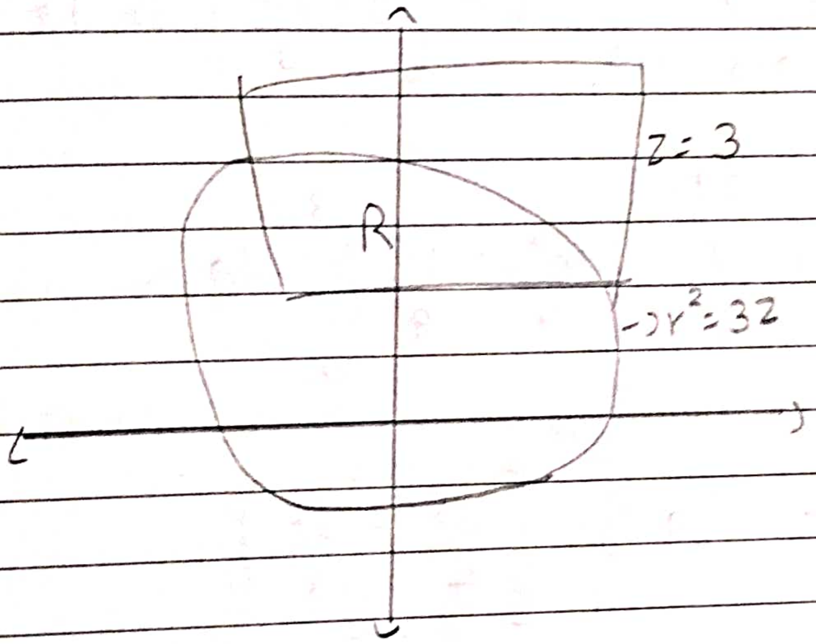
$$I = \frac{1}{3} \int_{-\pi/2}^{\pi/2} \left[\frac{2(4-r^2)^{3/2}}{8} \right]_0^{2\cos\theta} d\theta$$

$$I = \frac{1}{3} \int_{-\pi/2}^{\pi/2} [4-4\cos^2\theta]^{3/2} - (4)^{3/2} \, d\theta$$

$$I = \frac{-2}{15} \int_{-\pi/2}^{\pi/2} (2^5 \sin^5\theta - 25) \, d\theta$$

$$I = \frac{2^6 \pi}{15}$$

6 Evaluate $\iiint x^2+y^2 \, dx\,dy\,dz$ over the region bounded by paraboloid $x^2+y^2 = 3z$ and the plane $z=3$



Here, $x = r\cos\theta$, $y = r\sin\theta$ and $z=z$

$\to$ $x^2+y^2 = 3z$

$\therefore r^2 = 3(3)$

$\therefore r = 3$

$$z = \frac{r^2}{3} = 3$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{z=\frac{r^3}{3}}^{3} r^2 \cdot r \, dz \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \left[z \right]_{\frac{r^3}{3}}^{3} r^3 \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^3 \left(3 - \frac{r^3}{3} \right) dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \left[\frac{3r^4}{4} - \frac{r^6}{18} \right]_0^3 d\theta$$

$$I = \left[\theta \right]_0^{2\pi} \left[\frac{3^5}{4} - \frac{3^6}{18} \right]$$

$$I = \frac{81\pi}{2}$$

7 Evaluate $\iiint x^2y^2z^2\,dx\,dy\,dz$ over the region bounded by the surface $xy=4$, $xy=9$, $yz=1$, $yz=4$, $zx=25$, $zx=49$

Here, $xy=u$, $yz=v$, $xz=w$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix}$$

$$= 2xyz$$

$$\partial u\,\partial v\,\partial w = |J|\,\partial x\,\partial y\,\partial z$$

$$\therefore \partial x\,\partial y\,\partial z = \frac{1}{2xyz}\,\partial u\,\partial v\,\partial w$$

$$= \frac{1}{2\sqrt{uvw}}\,\partial u\,\partial v\,\partial w$$

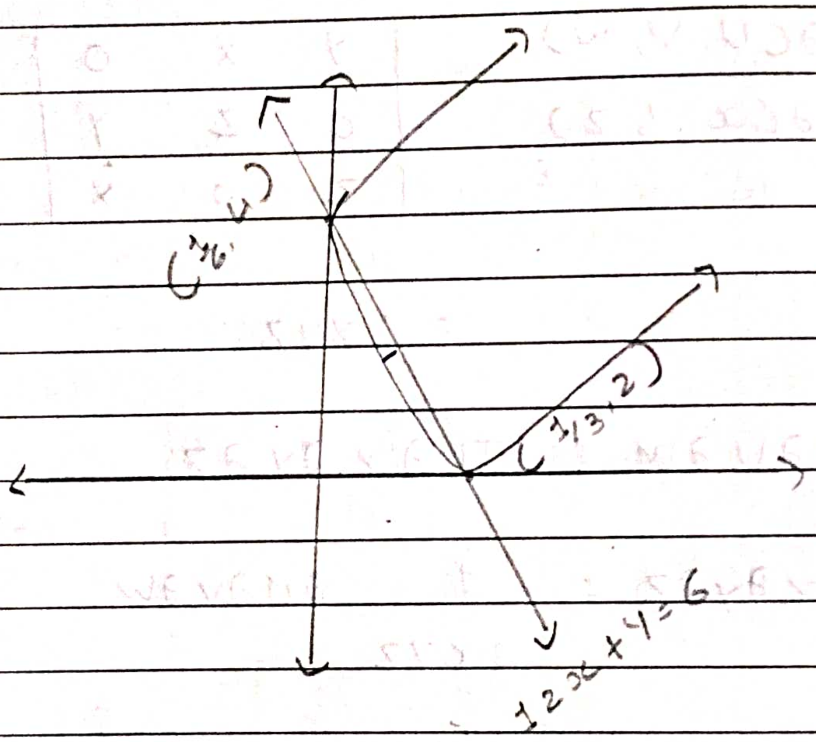
$$I = \int_{w=25}^{49}\int_{v=1}^{4}\int_{u=4}^{9} uvw \times \frac{1}{2\sqrt{uvw}}\,\partial w\,\partial v\,\partial w$$

$$I = \frac{4}{27}(343-125)(8-1)$$

$$I = \frac{115975}{27}$$

* Task 9 : Area as Double Integral

1 Find the area between the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$



$\Rightarrow$ Here, $12x + y = 6$

$\therefore y = 6 - 12x$

$\therefore 3x(6 - 12x) = 2$

$\therefore x = 1/3, \; 1/6$

$\therefore y = 2, \; 4$

$$\Rightarrow I = \int_{x=1/6}^{1/3} \int_{y=2/3x}^{6-12x} 1 \; dy \cdot dx$$

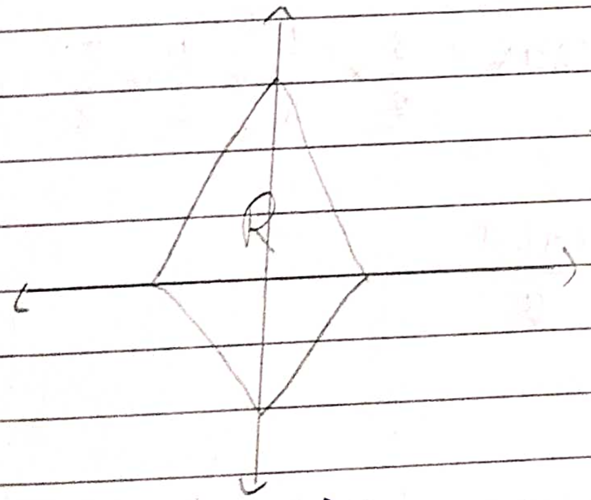
$$= \int_{1/6}^{1/3} 6 - 12x - \frac{2}{3x} \; dx$$

$$I = \left[6x - \frac{12x^2}{2} - \frac{2}{3} \log x \right]_{1/6}^{1/3}$$

$$I = \frac{1}{2} - \frac{2}{3} \log 2$$

2 Find the area bouned by the hypocycloid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3}$$



$$\therefore \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

$$\therefore y = b\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^{3/2}$$

$$\therefore I = 4\int_{x=0}^{a} b\left(1-\left(\frac{x}{a}\right)^{2/3}\right)^{3/2} dx$$

Let $x = a\cos^3 t$

$\therefore dx = 3a\cos^2 t(-\sin t)\,dt$

$x = 0 \rightarrow t = \pi/2$

$x = a \rightarrow t = 0$

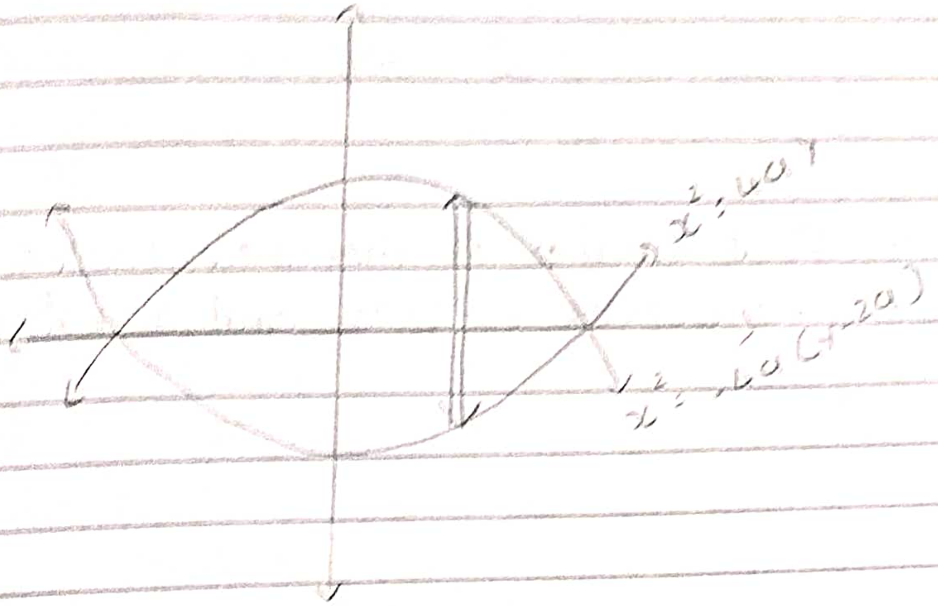
$$\therefore I = 4\int_{\pi/2}^{0} b(1-\cos^2 t)^{3/2}(-3a)\cos^2 t \sin t\, dt$$

$$\therefore I = 12ab\int_{0}^{\pi/2} \sin^4 t \cdot \cos^2 t \cdot dt$$

$$\therefore I = 12ab \times \frac{3}{6} \times \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore I = \frac{3ab\pi}{8}$$

3 Find the area bounded between the parabolas $x^2 = 4aY$ and $x^2 = -4a(Y-2a)$



$\therefore x^2 = 4ay$

$\therefore x^2 = -4a(y-2a)$

$\therefore 4ay = -4a(y-2a)$

$\therefore y = a$

$\therefore x^2 = \pm 2a$

$\therefore y^2 = \dfrac{x^2}{4a} = 2a - \dfrac{x^2}{4a}$

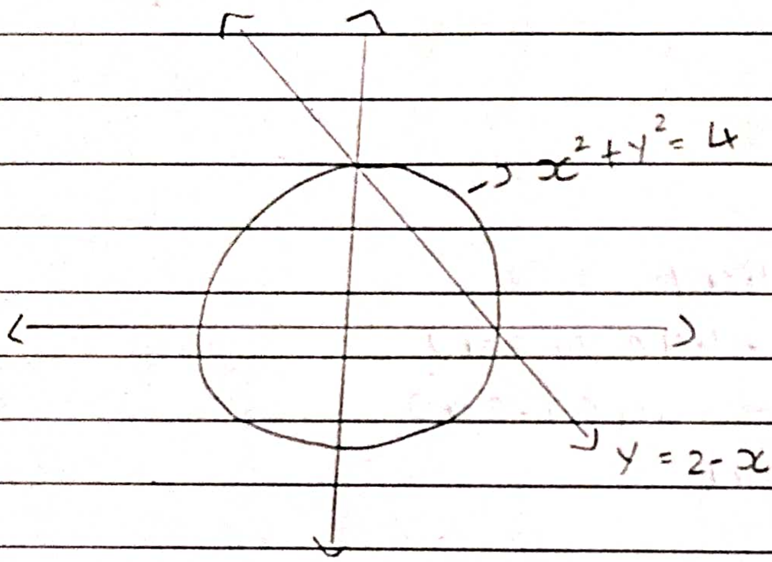
$$I = 2 \int_{x=0}^{2a} \int_{y=\frac{x^2}{4a}}^{2a-\frac{x^2}{4a}} 1 \cdot dy\,dx$$

$$I = 2 \int_{0}^{2a} 2a - \frac{x^2}{4a} - \frac{x^2}{4a} \; dx$$

$$I = 2 \left(2ax - \frac{x^3}{4a} \right)_0^{2a}$$

$$\therefore I = \frac{16a^2}{3}$$

4 Find smaller of the area enclosed by the curves $y = 2-x$ and $x^2+y^2=4$



$$\therefore x^2+(2-x)^2 = 4$$

$$\therefore x = 2 \quad \text{or} \quad x = 0$$

$$\therefore y = 0 \quad \text{or} \quad y = 2$$

$$I = \int_{x=0}^{2} \int_{y=2-x}^{\sqrt{4-x^2}} 1 \; dy \cdot dx$$

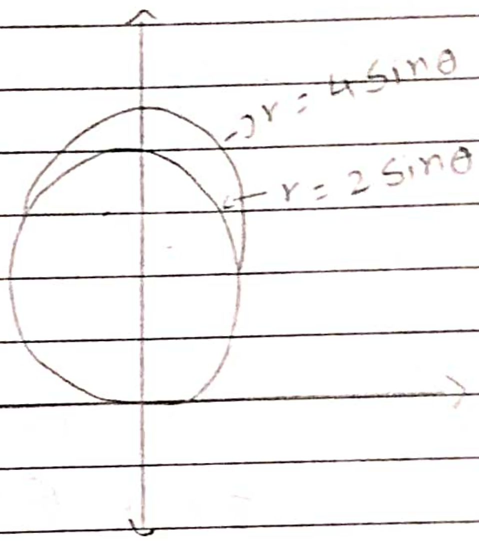
$$\therefore I = \int_0^2 \sqrt{4-x^2} - 2 + x \; dx$$

$$\therefore I = \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2} \right]_0^2$$

$\therefore I = \pi - 2$

5 Find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=2\sin\theta}^{4\sin\theta} r \, dr \cdot d\theta$$



$$\therefore I = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$\therefore I = \int_0^{\pi/2} 16\sin^2\theta - 4\sin^2\theta \cdot d\theta$$

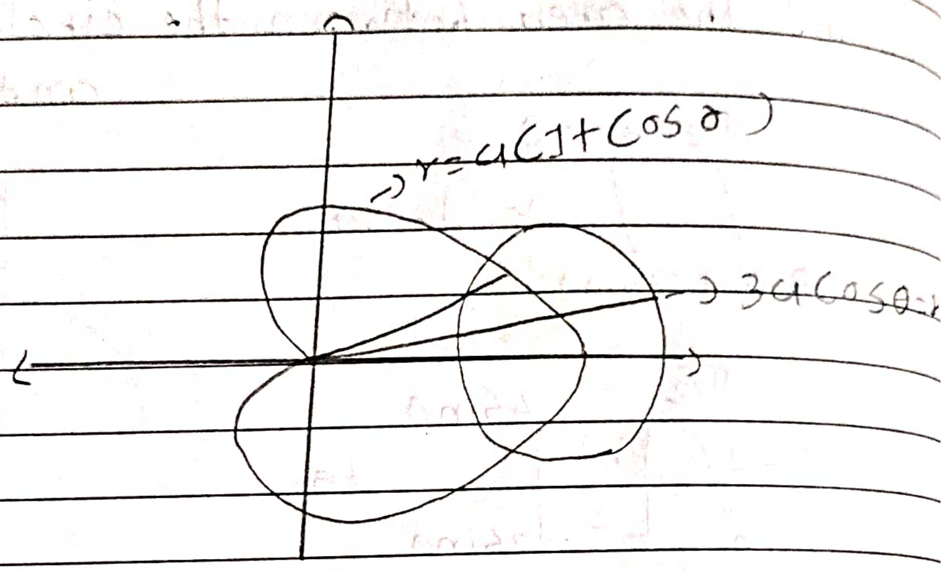
$$\therefore I = \int_0^{\pi/2} 12\sin^2\theta \cdot d\theta$$

$$\therefore I = 6\int_0^{\pi/2} 1 - \cos 2\theta \cdot d\theta$$

$$\therefore I = 6\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$\therefore I = 3\pi$

6 Find the area which lies inside the circle $r = 3a\cos\theta$ and $r = a(1+\cos\theta)$



$$\therefore I = 2 \int_0^{\pi/3} \int_{r=a+a\cos\theta}^{3a\cos\theta} r \, dr \cdot d\theta$$

$$\therefore I = 2 \int_0^{\pi/3} \left[\frac{r^2}{2}\right]_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$

$$\therefore I = a^2 \int_0^{\pi/3} a^2\cos^2\theta - a^2(1+\cos^2\theta)^2 \, d\theta$$

$$\therefore I = a^2 \int_0^{\pi/3} 4(1+\cos 2\theta - 1 - 2\cos\theta] \, d\theta$$

$$\therefore I = a^2 \left[3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right]_0^{\pi/3}$$

$\therefore I = a^2\pi$

7 Find the area common to cardioid $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$

$\therefore a(1+\cos\theta) = a(1-\cos\theta)$

$\therefore \cos\theta = 0$

$\therefore \theta = \pi/2$

$$I = 4\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$I = 4\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

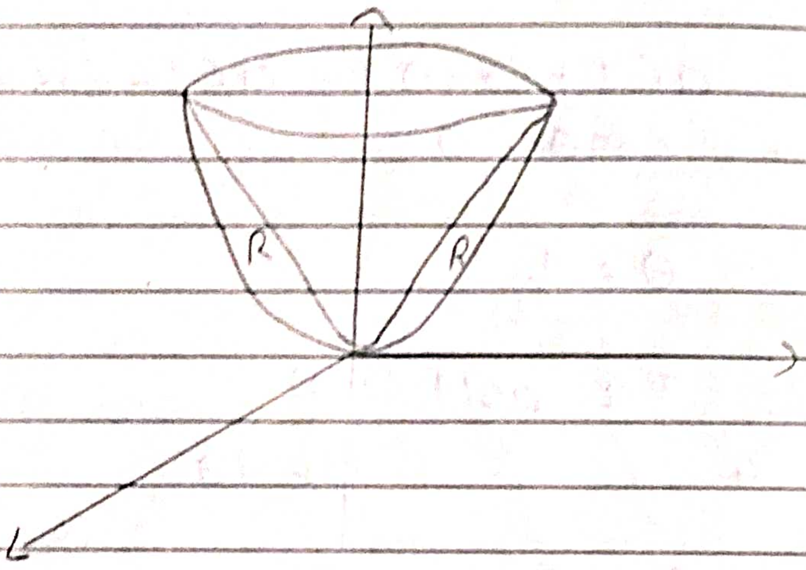
$$I = 2a^2\int_0^{\pi/2} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta$$

$$\therefore I = 2a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$\therefore I = 2a^2\left(\frac{3\pi}{4} - 2\right)$$

* Task 10: Volume as Triple Integral.

1 Find the Volume bounded by the cone $x^2+y^2=z^2$ and paraboloid $x^2+y^2=z$



=> $x^2+y^2=z^2$, $x^2+y^2=z$

∴ $r^2=z^2$ ∴ $r^2=z$

$$V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=r^2}^{r} r \, dz \, dr \, d\theta$$

$$\therefore V = 4\int_{0}^{\pi/2}\int_{0}^{1} r\cdot(z)_{r^2}^{r} \, dr \, d\theta$$

$$\therefore V = 4\int_{0}^{\pi/2}\int_{0}^{1} r(r^2-r^2) \, dr \cdot d\theta$$

$$\therefore V = 4\int_0^{\pi/2}\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta$$

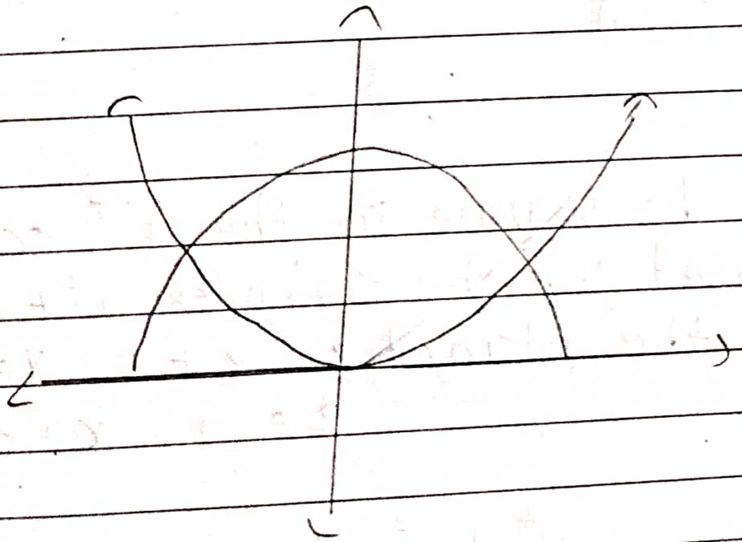
$$\therefore V = 4\int_0^{\pi/2} \frac{1}{3} - \frac{1}{4}\, d\theta$$

$$\therefore V = 4\left[\theta\right]_0^{\pi/2}\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\therefore V = 4\cdot\frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\therefore V = \frac{\pi}{6}$$

2 Find the volume of the solid bounded by the parabolid $x^2+y^2 = z$ and $z = 4-3(x^2+y^2)$



$\Rightarrow$ $x^2+y^2 = z$ $\qquad$ $z = 4-3r^2$

$\therefore z = r^2$

$\therefore z \to r^2 \to 4-3r^2$

$$\therefore V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{z=r^2}^{4-3r^2} r \, dz \, dr \, d\theta$$

$$\therefore V = 4 \int_{0}^{\pi/2} \int_{0}^{1} r(4-3r^2-r^2) \, dr \, d\theta$$

$$\therefore V = 4 \int_{0}^{\pi/2} \left[\frac{4r^2}{2} - \frac{4r^4}{4} \right]_0^1 d\theta$$

$$\therefore V = 4 \int_{0}^{\pi/2} 1 2 \, d\theta$$

$$\therefore V = 4 \left[1\theta \right]_0^{\pi/2}$$

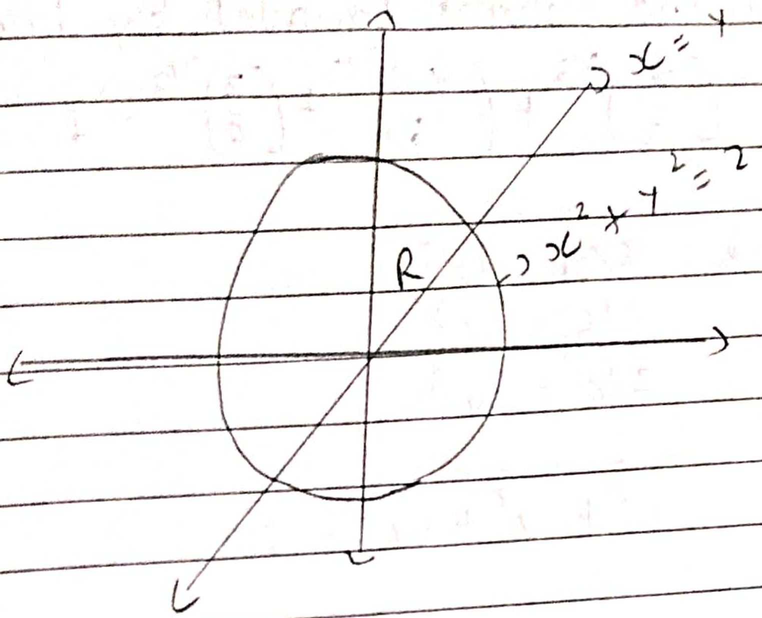
$$\therefore V = 2\pi$$

3 Find the volume in the 1^{st} octant bounded by the cylinder $x^2+y^2=2$ and the planed $z = x+y$, $y = x$, $z = 0$, $x = 0$

$$\therefore x^2 + y^2 = 2$$

$$\therefore 2x^2 = 2$$

$$\therefore x = \pm 1, \quad y = \pm 1$$



$$V = \int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \int_{z=0}^{x+y} 1 \, dz \cdot dy \cdot dx$$

$$\therefore V = \int_{0}^{1} \int_{x}^{\sqrt{2-x^2}} x + y \, dy \, dx$$

$$\therefore V = \int_{0}^{1} \left[xy + \frac{y^2}{2} \right]_{x}^{\sqrt{2-x^2}} dx$$

$$\therefore V = \int_{0}^{1} \left(x\sqrt{2-x^2} - x \right) + \frac{1}{2}\left(2 - x^2 - x^2\right) dx$$

$$\therefore V = \frac{2\sqrt{2}}{3}$$

4 Find the volume bounded by the solid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Here, $x/a = u^3$
$y/b = v^3$
$z/c = w^3$

$\therefore \quad u^2 + v^2 + w^2 = 1$

Here, $u = r\sin\theta\cos\phi$
$v = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\Rightarrow J = \frac{\partial(x,y,z)}{\partial(u,w,w)} = \begin{vmatrix} 3a\,y^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3z^2w \end{vmatrix}$$

$$= 27\,a^3 v^2 u^2 w^2$$

$$\Rightarrow V = 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} 27a^3 v^2 u^2 w^2 \, du\,dw\,dw$$

$$V = 216abc \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\theta \cdot r^2 \sin^2\theta \sin^2\phi \, r^2 \cos^2\theta \, dr\,d\theta\,d\phi$$

$$\therefore V = \frac{4\pi}{35} abc$$