Linear Transformation.

* Task : 1 Linear Transformation and Linear Operator.

(i) Determine whether the following function is a Linear Transformation.

A $T : R^2 \to R^3$, define by $T(x, y) = (x-y, x+2y, 6y)$

Let, $U = (x_1, y_1)$ and $V = (x_2, y_2)$

(1) $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$T(U+V) = (x_1 + x_2 - y_1 - y_2, x_1 + x_2 + 2y_1 + 2y_2, 6y_1 + 6y_2)$ — ①

$\rightarrow$ $T(U) + T(V) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1 - y_1, x_1 + 2y, 6y_1) + (x_2 - y_2, x_2 + 2y_2, 6y_2)$

$= (x_1 + x_2 - y_1 - y_2, x_1 + 2y_1 + x_2 + 2y_2, 6y_1 + 6y_2)$ — ②

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by eq^n 1 and 2

$\therefore T(U+V) = T(U) + T(V)$ — a

-> $KU = K(x_1, y_1)$

$= (Kx_1, Ky_1)$

$T(KU) = (Kx_1 - Ky_1, Kx_1 + 2Ky_1, 6Ky_1)$

$= K(x_1 - y_1, x_1 + 2y_1, 6y_1)$ — 1

-> $KT(U) = KT(x_1, y_1)$

$= K(x_1 - y_1, x_1 + 2y_1, 6y_1)$ — 2

$KT(U) = T(KU)$ — b

By eq^n a and b

T is linear transformation.

B $T: R^2 - R^2$, define by $T(x, y) = (x+y, xy)$

Let, $U = (x_1, y_1)$ and $V = (x_2, y_2)$

(1) $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$\rightarrow T(U+V) = (x_1+x_2+y_1+y_2,\ (x_1+x_2)(y_1+y_2))$ — (1)

$\rightarrow T(U) = (x_1+y_1,\ x_1y_1)$

$T(V) = (x_2+y_2,\ x_2y_2)$

$T(U)+T(V) = (x_1+y_1+x_2+y_2,\ x_1y_1+x_2y_2)$ — (2)

By eq^n 1 and 2

$T(U+V) \neq T(U)+T(V)$

So, that T is not linear Transformation.

($T: R^2 \rightarrow R^3$, define by $T(x,y) = (3x, x+y, 6)$

Let $U = (x_1, y_1)$ and $V = (x_2, y_2) \in R^2$

(1) $U+V = (x_1, y_1) + (x_2, y_2)$

$= (x_1+x_2,\ y_1+y_2)$

$\rightarrow T(U+V) = (3(x_1+x_2),\ x_1+x_2+y_1+y_2,\ 6)$

$= (3x_1+3x_2,\ x_1+x_2+y_1+y_2,\ 6)$ — (1)

$\rightarrow T(U) = (3x_1,\ x_1+y_1,\ 6)$

$T(V) = (3x_2,\ x_2+y_2,\ 6)$

$T(U) + T(V) = (3x_1 + 3x_2, x_1 + x_2 + y_1 + y_2 ; 12)$ — 2

Here, eq^n $1 \neq 2$

$T(U+V) \neq T(U) + T(V)$

So, that T is not linear transformation.

D) $T : M_{22} \to R$, define by $T(A) = tr(A)$

Let, $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $V = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_{22}$

1 → $U + V = \begin{bmatrix} a + a_1 & b + b_1 \\ c + c_1 & d + d_1 \end{bmatrix}$

$T(U+V) = a + a_1 + d + d_1$ — 1

→ $T(U) = a + d$

$T(V) = a_1 + d_1$

$T(U) + T(V) = a + d + a_1 + d_1$ — 2

Here, eq^n $1 = 2$

$\therefore T(U+V) = T(U) + T(V)$

(2) For Scaler K_1

$\cancel{K(U) = K(x_1, y_1)}$

$\cancel{= (kx_1, ky_1)}$

$\cancel{T(K(U)) =}$

$$K(U) = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

$T\,K(U) = ka + kd$ — (1)

$\rightarrow K \cdot T(U) = K \cdot (a + d)$

$= ka + kd$ — (2)

by eq^n $1 = 2$

$\therefore T\,K(U) = K \cdot T(U)$

So, that T is Linear Transformation.

E $T : P \rightarrow P$, define by $T(P(x)) = 1 + 2x + 3x^2 + 4x^3$

Let $U = 1 + 2x_1 + 3x_1^2 + 4x_1^3$,
$V = 1 + 2x_2 + 3x_2^2 + 4x_2^3$

$\rightarrow \cancel{U + V = 2 + 2(x}$

(ii) Find the standart matrix of a rotation of 45° about the Y-axis, followed by a dilation with the factor $K = 2\sqrt{2}$ of linear operators on R^3.

$\Rightarrow$ Let, T is rotation about the 45° Y axis through angle of R^3.

$\rightarrow T(x, y, z) = (x\cos\theta + z\sin\theta,\ y,\ -x\sin\theta + z\cos\theta)$

Standart matrix of T $= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

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For $\theta = 45^\circ$

$$T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

-> Let, $T_1(x, y, z) = (kx, ky, kz)$

$$T_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

For $k = 2\sqrt{2}$

$$T_1 = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

-> Linear transformation,

$$T_0 = T \cdot T_1$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2\sqrt{2} & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

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(iii) Find the standard matrix of a rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction with the factor $k = 1/2$ of linear operators on R^3.

→ matrix of T is rotation of 30° about the x axis,

$T(x, y, z) = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

For $\theta = 30°$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$$

→ matrix of T_1 is rotate about the z-axis,

$T_1(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$

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$$T_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For $\theta = 30$

$$T_1 = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

=> ~~Let T_2 Ex~~

-> Let $T_2(x, y, z) = (kx, ky, kz)$

$$T_2 = \begin{bmatrix} kx & 0 & 0 \\ 0 & ky & 0 \\ 0 & 0 & kz \end{bmatrix}$$

For $k = 1/2$

$$T_2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

-> Linear Transformation

$$T_0 = T \cdot T_1 \cdot T_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & -\sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/4 & -1/4 & 0 \\ 1/4 & \sqrt{3}/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$T_0 = \begin{bmatrix} \sqrt{3}/4 & -\sqrt{3}/8 & 1/8 \\ 1/4 & 3/8 & \sqrt{3}/8 \\ 0 & 1/4 & \sqrt{3}/4 \end{bmatrix}$$

iv Consider the basis $B = \{(-2,1),(1,3)\}$ and Let $T: R^2 \to R^3$ be the linear transformation such that $T(-2,1) = (-1,2,0)$ $T(1,3) = (0,-3,5)$. Then Find the formula for $T(x_1, x_2)$ and use this formula to find $T(2,-3)$

Here, given $v_1 = (-2,1)$

$T(v_1) = (-1,2,0)$

$v_2 = (1,3)$

$T(v_2) = (0,-3,5)$

Let, $v = k_1 v_1 + k_2 v_2$

$\therefore (x_1, x_2) = k_1(-2,1) + k_2(1,3)$

$\therefore (x_1, x_2) = (-2k_1, k_1) + (k_2, 3k_2)$

$\therefore (x_1, x_2) = (-2k_1 + k_2, k_1 + 3k_2)$

$\therefore -2k_1 + k_2 = x_1$ — ①

$\therefore k_1 + 3k_2 = x_2$ — ② $\times 2$

$$\therefore\ 2K_1 + 6K_2 = 2x_2$$
$$-2K_1 + K_2 = x_1$$
$$+ \qquad - \qquad -$$

$$7K_2 = 2x_2 + x_1$$

$$\boxed{K_2 = \frac{2x_2 + x_1}{7}}$$

put K_2 value in eq^n -1

$$\therefore\ -2K_1 + \frac{2x_2 + x_1}{7} = x_1$$

$$\therefore\ -2K_1 = x_1 - \frac{2x_2 + x_1}{7}$$

$$\therefore\ -2K_1 = \frac{7x_1 - 2x_2 + x_1}{7}$$

$$\therefore\ -2K_1 = \frac{6x_1 - 2x_2}{7}$$

$$\therefore\ \boxed{K_1 = \frac{x_2 - 3x_1}{7}}$$

→ $V = K_1V_1 + K_2V_2$

$$\therefore\ T(V) = K_1\,T(V_1) + K_2\,T(V_2)$$

$$\therefore\ T(x_1, x_2) = K_1(-1, 2, 0) + K_2(0, -3, 5)$$

$$\therefore T(x_1, x_2) = \frac{x_2 - 3x_1}{7}(-1, 2, 0) + \frac{2x_2 + x_1}{7}(0, -3, 5)$$

$$\therefore T(x_1, x_2) = \frac{-x_2 + 3x_1}{7} + \frac{2x_2}{7} - \frac{6x_1}{7}$$

$$-\frac{6x_2}{7} - \frac{3x_1}{7} + \frac{10x_2}{7} + \frac{5x_1}{7}$$

$$= \frac{-x_2 + 2x_2 + 10x_2 - 6x_2}{7} +$$

$$\frac{(3x_1 - 6x_1 - 3x_1 + 5x_1)}{7}$$

$$= \frac{5x_2}{7} - \frac{x_1}{7}$$

$$\therefore T(x_1, x_2) = \left(\frac{-x_2 + 3x_1}{7}, \frac{2x_2 - 6x_1}{7}, 0\right)$$

$$+ \left(0, \frac{-6x_2 - 3x_1}{7}, \frac{10x_2 + 5x_1}{7}\right)$$

$$= \left(\frac{-x_2 + 3x_1}{7}, \frac{-4x_2 - 9x_1}{7}, \frac{10x_2 + 5x_1}{7}\right)$$

$\rightarrow$ $T(2,-3) = \left(\frac{3(2)-(-3)}{7}, \frac{-4(-3)-9(2)}{7}, \frac{10(-3)+5(2)}{7} \right)$

$= \left(\frac{9}{7}, -\frac{6}{7}, \frac{-20}{7} \right)$

V Consider the linear transformation $T: P_2 \rightarrow P_2$ such that $T(1) = 1+x$, $T(x) = 1+x$, $T(x^2) = 5+3x-2x^2$. then Find the formula For $T(a_0 + a_1 x + a_2 x^2)$ and use it to Find $T(2+3x-5x^2)$.

Here, given, $V_1 = 1 \rightarrow T(V_1) = 1+x$
$V_2 = x \rightarrow T(V_2) = 1+x$
$V_3 = x^2 \rightarrow T(V_3) = 5+3x-2x^2$

Let, $V = V_1 K_1 + K_2 V_2 + K_3 V_3$

$\therefore (a_0 + a_1 x + a_2 x^2) = K_1 + x K_2 + x^2 K_3$

Hence, $K_1 = a_0$
$K_2 = a_1$
$K_3 = a_2$

$\rightarrow$ $T(V) = K_1 T(V_1) + K_2 (T(V_2)) + K_3 T(V_3)$

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$\rightarrow T(a_0 + a_1 x + a_2 x^2) = a_0(1+x) + a_1(1+x) + a_2(5+3x-2x^2)$

$= a_0 + a_0 x + a_1 + a_1 x + 5a_2 + 3x a_2 - 2x^2 a_2$

$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1 + 5a_2) + (a_0 x + a_1 x + 3x a_2) + x^2(-2a_2)$

$\rightarrow T(2 + 3x - 5x^2) = (a_0 + a_1 + 5a_2) + x(a_0 + a_1 + 3a_2) + x^2(-2a_2)$

than, $-2a_2 = -5$

$\therefore \boxed{a_2 = 5/2}$

$\rightarrow a_0 + a_1 + 5a_2 = 2$

$\therefore a_0 + a_1 + \frac{25}{2} = 2$

$\therefore a_0 + a_1 = 2 - \frac{25}{2} = \frac{-21}{2}$ — 1

$\rightarrow a_0 + a_1 + 3a_2 = 3$

$\therefore a_0 + a_1 + \frac{15}{2} = 3 \rightarrow a_0 + a_1 = \frac{-9}{2}$ — 2

From eq^n 1 and 2,

$$a_0 + a_1 = \frac{-21}{2}$$

$$\cancel{a_0} + a_1 = \frac{-9}{2} - a_0$$

$$\therefore \cancel{a_0} - \frac{9}{2} - a_0 = \frac{-21}{2}$$

$\therefore$

* Task : 2 Kernal and Range of Linear Transformation and Rank-nullity Theorem.

(i) Find the kernal and range of the following Linear transsformation.

a $T: R^2 \to R^2$, define by
$T(x, y) = (2x - y, -8x + 4y)$

→ System of eq^n $AX = 0$

$$\therefore \quad 2x - y = 0$$
$$-8x + 4y = 0$$

Augmented matrix,

$$[A \mid B] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -8 & 4 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{cc|c} -8 & 4 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_1 \to R_1/-8$$

$$= \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

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$$= \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

→ Range(T) = (leading 1 in columne to the corresponding element).

$$= \left\{ \begin{bmatrix} 2 \\ -8 \end{bmatrix} \right\}$$

→ $2x - y = 0$

Let $y = t$

$\therefore 2x = t$

$\therefore x = \frac{t}{2}$

→ $\text{Ker}(T) = \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

6 $T: R^4 \to R^3$ define by
$T(x, y, z, w) = (4x + y - 2z - 3w,$
$2x + y + z - 4w,$
$6x - 3z + 9w)$

→ System of equation $AX = 0$

$\therefore 4x + y - 2z - 3w = 0$

$\therefore 2x + y + z - 4w = 0$

$\therefore 6x + 0y - 3z + 9w = 0$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -3 & 9 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$= \left[\begin{array}{cccc|c} 2 & 0 & -3 & 1 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -3 & 9 & 0 \end{array}\right]$$

$$R_1 \to R_1/2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -3/2 & 1/2 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -3 & 9 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 6R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -3/2 & 1/2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 3 & 6 & 0 \end{array}\right]$$

$$R_3 \to R_3/3$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -3/2 & 1/2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\therefore x_1 - \frac{3}{2}x_3 + \frac{1}{2}x_4 = 0$$

$$\therefore x_2 + 4x_3 - 5x_4 = 0$$

$$\therefore x_3 + 2x_4 = 0$$

$$\therefore x_3 = -2x_4$$

$\rightarrow$ $Range(T) = \left\{ \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \right\}$

(ii) Let T be multiplication by the matrix $A = \begin{bmatrix} 1 & 4 & -5 & 0 & 9 \\ 3 & -2 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 3 & 5 & 1 & 8 \end{bmatrix}$ then Find the basis of Kernal and range of T.

$\rightarrow$ Augmented matrix, $AX = 0$

$$[A|B] = \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 3 & -2 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 2 & 3 & 5 & 1 & 8 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 + R_1$
$R_4 \rightarrow R_4 - 2R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 0 & -14 & -14 & 0 & -28 & 0 \\ 0 & 4 & 4 & 0 & 8 & 0 \\ 0 & -5 & -5 & 1 & -10 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$= \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 0 & -5 & -5 & 1 & -10 & 0 \\ 0 & 4 & 4 & 0 & 8 & 0 \\ 0 & -14 & -14 & 0 & -28 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 / 4$
$R_4 \rightarrow R_4 / +14$

$$= \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 0 & -5 & -5 & 1 & -10 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & -2 & 0 \end{array}\right]$$

$R_4 \to R_4 + R_3$
$R_2 \to R_2 + 5R_3$

$$= \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$= \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 9 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\to$ Let $x_4 = 0$

$x_2 + x_3 + 2x_5 = 0$
$x_1 + 4x_2 + 5x_3 + 9x_5 = 0$

Let $x_3 = t_2$, $x_5 = t_1$

$\therefore x_2 = -t_2 - 2t_1$

$\therefore x_1 = -(t_1 + t_2)$

$$\to X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t_1 - t_2 \\ -2t_1 - t_2 \\ t_2 \\ 0 \\ t_1 \end{bmatrix}$$

$$X = t_1 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ $Ker(T) = \left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\rightarrow$ $Range(T) = \{ x_1(1, 3, -1, 2) + x_2(4, -2, 0, 3) + x_4(0, 0, 0, 1) \}$

(iv) Let $T : M_{22} \rightarrow M_{22}$ be the linear transformation befind by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$$

Find the basis for $ker(T)$ and $R(T)$.

$\rightarrow$ Here, $a+b=0$

$b+c=0$

$a+d=0$

$b+d=0$

$\rightarrow$ Augmented matrix $AX=0$

$\rightarrow$ $[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$

$R_3 \rightarrow R_3 - R_1$

$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$

$R_4 \rightarrow R_4 + R_3$

$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$

$R_3 \rightarrow R_3 + R_2$
$R_4 \rightarrow \frac{R_4}{2}$

$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$\rightarrow$ From matrix, $c + d = 0$ and $d = 0$
$b + c = 0$
$a + b = 0$

$\rightarrow$ $Ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$\rightarrow$ $T\left(\begin{bmatrix} a & b \\ c & a \end{bmatrix} \right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$

$$= a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$+ c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow$ $R(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(V) Verify the Rank-nullity theorem of the linear transformation,

$T: R^4 \rightarrow R^3$, $T(x, y, z, w) = (x - y + z + w,$
$x + 2z - w,$
$x + y + 3z - 3w)$

$\rightarrow$ System of equation,

$$x - y + z + w = 0$$
$$x + 2z - w = 0$$
$$x + y + 3z - 3w = 0$$

→ Augmented matrix, $AX = 0$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 2 & 2 & -4 & 0 \end{array}\right]$$

R

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 3 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ $x_1 - x_2 + x_3 + x_4 = 0$
$x_2 + x_3 - 2x_4 = 0$

Let $t_2 = x_4$ and $x_3 = t_1$

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$\therefore x_2 = 2t_2 - t_1$
$\therefore x_1 = t_2 - 2t_1$

$\to X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_2 - 2t_1 \\ 2t_2 - t_1 \\ t_1 \\ t_2 \end{bmatrix}$

$= t_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\to \ker(T) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\to$ Rank of matrix is $= 2$

$\to R(T) + n(T) = \dim(V)$

$\therefore 2 + 2 = \dim(V)$

$\therefore \dim(R_4) = 4$

Hence, dimension theorem is Verified.

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(vi) Verify the Rank-nullity theorem of the linear transformation,

$T: P_3 \to R_2$ defined by

$T(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$

→ Here, solution space of $T(x) = 0$

$\therefore \ 3ax^2 + 2bx + c = 0$

$\therefore \ 3ax^2 + 2bx + c = 0x^2 + 0x + 0$

$\therefore \ 3a = 0, \ 2b = 0, \ c = 0$

$\therefore \ a = 0, \ b = 0$ and $c = 0$

→ $\ker(T) = 0$

$\dim(\ker(T)) = n(T) = 0$

→ $R(T) + n(T) = \dim(V)$

$\dim(V) = 0$

→ Hence, dimension theorem is verify.

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(iii) Find the basis of kernal and range of the following linear transformation

$T: P_2 \to P_2$ defined by

$T(a_0 + a_1 x + a_2 x^2) = 2a_0 + a_1(x+2)$
$+ a_2(x+2)^2$

→ Here, solution space $T(x) = 0$

$\therefore 2a_0 + a_1(x+2) + a_2(x+2)^2 = 0$

$\therefore 2a_0 = 0$
$\therefore a_1(x+2) = 0$
$\therefore a_2(x^2+4x+4) = 0$

→ Augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3/2$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\rightarrow$ Here, $R(T) = \{2, (x+2), (x+2)^2\}$

$\rightarrow$ $Ker(T) = 0$

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* Task : 3: One to One, Onto and Inverse Linear Transformation.

(i) Determine whether the following linear transformation are one to one or onto.

(a) $T: R^2 \to R^3$ define by $T(x, y) = (x, y, x-y)$

$\Rightarrow$ For one to one

Here, $T(x, y) = 0$

$\therefore (x, y, x-y) = 0$

$\therefore$

$x = 0$

$y = 0$

$x - y = 0$

$\to$ Sol^n $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \text{Ker}(T) = 0$

$\therefore T: R^2 \to R^3$ is one to one L.T.

$\Rightarrow$ For onto,

Let, $V = (a, b) \in R^2$

$\therefore T(x, y) = (a, b)$

$\therefore (x, y, x-y) = (a, b)$

$\therefore x = a$
$y = b$
$\therefore x - y = 0 \notin R^3$

So, that, $T: R^2 \to R^3$ is not Onto L.T.

b $T: R^2 \to R^3$ define by
$T(x, y) = (x-y, y-x, 2x-2y)$

$\Rightarrow$ For one to one,

$T(x, y) = 0$

$\therefore (x-y, y-x, 2x-2y) = 0$

$\therefore x - y = 0$
$y - x = 0$
$2x - 2y = 0$

$\rightarrow$ $A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 + 2R_1$

$\therefore A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let, $y = t$
$\therefore x - y = 0$
$\therefore x = t$

$\rightarrow$ $Sol^n = X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore Ker(T) \neq 0$

So, that $T: R^2 \rightarrow R^3$ is not one to one L.T.

$\Rightarrow$ For onto,

$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$

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$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\text{Rank}(A) = 1$

Here matrix is 3×2 form.

$\text{Rank}(A) \neq 3$

$\therefore T: R^2 \to R^3$ is not onto.

c $T: R^3 \to R^2$ define by
$T(x, y, z) = (x + y + z$
$y + x,$

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Here, matrix is 2×3 form.

$R_2 \to R_2 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

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Rank(A) = 2

$\therefore$ Rank(A) = 2 = m
$\therefore$ Rank (A) $\neq$ 3 = n

So, $T: R^3 \to R^2$ is one to one and not onto function.

d $T: R^4 \to R^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, \; 2x_1 + x_2 + x_3 - 4x_4, \; 6x_1 + 0x_2 - 9x_3 + 9x_4)$$

$$A = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1$
$R_3 \to 2R_3 - 3R_1$

$$\sim \left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & -3 & -12 & 27 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$\sim \left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & -12 & 12 & 0 \end{array}\right]$$

$Rank(A) = 3$

Here, matrix form is 3×4.

$Rank(A) = 3 = n$

So, that L.T is onto.

$Rank(A) = 3 \neq m \neq 4$

So, that L.T is not one to one.

(ii) Determine inverse linear transformation of the following linear transformation.

(a) $T: R^3 \to R^3$ define by

$T(x, y, z) = (x - y + z, 2x - z, 2x + 3y)$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

$$\rightarrow [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow 2R_3 - 5R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & 0 & 11 & 6 & -5 & 1 \end{array}\right]$$

$R_2 \rightarrow 11R_2 + 3R_3$
$R_1 \rightarrow 11R_1 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 5 & 5 & -1 \\ 0 & 2 & 0 & -4 & -4 & 3 \\ 0 & 0 & 11 & 6 & -5 & 1 \end{array}\right]$$

$R_1 \rightarrow 2R_1 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 6 & 6 & 1 \\ 0 & 2 & 0 & -4 & -4 & 3 \\ 0 & 0 & 1 & 6 & -5 & 1 \end{array}\right]$$

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$R_1 \to R_1/2$
$R_2 \to R_2/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 1/2 \\ 0 & 1 & 0 & -2 & -2 & 3/2 \\ 0 & 0 & 1 & 6 & -5 & 1 \end{array}\right]$$

$\therefore 3x + 3y + \frac{1}{2}z = 0$

$\therefore -2x - 2y + \frac{3}{2}z = 0$

$\therefore 6x - 5y + z = 0$

$\to T^{-1}(x, y, z) = \left(3x + 3y + \frac{1}{2}z,\ -2x - 2y + \frac{3}{2}z,\ 6x - 5y + z\right)$

(b) $T : R^3 \to R^3$ define by

$T(x, y, z) = (x + 4y - z,\ x + 2y + z,\ -x + y)$

$$\Rightarrow A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 + R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 5 & -1 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow 2R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 8 & -3 & 5 & 2 \end{array}\right]$$

$R_1 \rightarrow 8R_1 + R_3$

$R_2 \rightarrow 4R_2 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 2 & 0 & 5 & 5 & 2 \\ 0 & -8 & 0 & -1 & -1 & -2 \\ 0 & 0 & 8 & -3 & 5 & 2 \end{array}\right]$$

$R_1 \rightarrow R_1 + 4R_2$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 1 & 1 & -6 \\ 0 & -8 & 0 & -1 & -1 & -2 \\ 0 & 0 & 8 & -3 & 5 & 2 \end{array}\right]$$

$R_1 \to R_1/8$
$R_2 \to -R_2/8$
$R_3 \to R_3/8$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/8 & 1/8 & -3/4 \\ 0 & 1 & 0 & 1/8 & 1/8 & 1/4 \\ 0 & 0 & 1 & -3/8 & 5/8 & 1/4 \end{array}\right]$$

$$\frac{1}{8}x + \frac{1}{8}y \frac{-3}{4}z = 0$$

$$\frac{1}{8}x + \frac{1}{8}y + \frac{1}{4}z = 0$$

$$\frac{-3}{8}x + \frac{5}{8}y + \frac{1}{4}z = 0$$

$$\to T^{-1}(x, y, z) = \left(\frac{1}{8}x + \frac{1}{8}y \frac{-3}{4}z,\ \frac{1}{8}x + \frac{1}{8}y + \frac{1}{4}z,\ \frac{-3}{8}x + \frac{5}{8}y + \frac{1}{4}z\right)$$

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* Task : 4 Change of Bases

1 Let $T: R^2 \to R^2$ be linear operator define by $T(x_1, x_2) = (5x_1 - x_2, 2x_1 + x_2)$ and $B = \{(1,0),(0,1)\}$, $B' = \{(1,4),(2,7)\}$ be two bases then Find transition P from B to B' and transition Q from B' to B.

$\Rightarrow$ The transition P from B to B'

$$(1, 4) = k_1(1,0) + k_2(0,1)$$

$$\therefore (k_1, k_2) = (1, 4)$$

$\to$ $Sol^n = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$(2, 7) = k_1(1,0) + k_2(0,1)$$

$$\therefore (k_1, k_2) = (2, 7)$$

$\to$ $Sol^n = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$\Rightarrow$ transition $P = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

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=> The transition matrix from B' to B

$\therefore K_1(1,4) + K_2(2,7) = (1,0)$

$\therefore K_1 + 2K_2 = 1$
$4K_1 + 7K_2 = 0$

$\therefore K_1 = -7$
$K_2 = 4$

-> $K_1(1,4) + K_2(2,7) = (0,1)$

$\therefore K_1 + 2K_2 = 0$
$\therefore 4K_1 + 7K_2 = 1$

$\therefore K_1 = 2$
$K_2 = -1$

=> transition matrix B' to $B = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$

2 Let $T: R^2 \to R^2$ be linear operator define by $T(x_1, x_2) = (x_1 - 2x_2, -x_2)$ and $B = \{(1,0), (0,1)\}$, $B' = \{(2,1), (-3,4)\}$ be two bases then find $[T]_B$ and $[T']_B$.

$P_1 = (1,0)$ and $P_2 = (0,1)$
$W_1 = (2,1)$ and $W_2 = (-3,4)$

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$\rightarrow$ $T(P_1) = k_1 P_1 + k_2 P_2$
$T(1,0) = k_1(1,0) + k_2(0,1)$
$\therefore (k_1, k_2) = (1, 0)$

$\rightarrow$ $T(P_2) = k_1 P_1 + k_2 P_2$
$\therefore T(2,1) = k_1(1,0) + k_2(0,1)$
$\therefore (k_1, k_2) = (2, 1)$

$\rightarrow$ $T[B] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ Expressing as linear combination of P_1 and P_2

$k_1 P_1 + k_2 P_2 = a_{11}$
$\therefore k_1(1,0) + k_2(0,1) = (2,1)$

$\therefore (k_1, k_2) = (2, 1)$

$\therefore k_1 = 2, \quad k_2 = 1$

$\Rightarrow$ $[a_{11}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow$ Expressing as linear combination of P_1 and P_2

$\therefore k_1 P_1 + k_2 P_2 = a_{12}$
$\therefore k_1(1,0) + k_2(0,1) = (3,4)$
$\therefore (k_1, k_2) = (3, 4)$

$$K_1 = -3 \quad , \quad K_2 = 4$$

$\Rightarrow \quad [a_{12}]_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$\Rightarrow$ transition matrix $P = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$

$$P^{-1} = \frac{1}{11}\begin{bmatrix} 1/2 & 3 \\ -1 & 2 \end{bmatrix}$$

$\Rightarrow \quad [T]^{B^{-1}} = P^{-1}\,[T]_B\,P$

$$= \frac{1}{11}\begin{bmatrix} 4 & 11 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -3 & -56 \\ -2 & 3 \end{bmatrix}$$

$\cancel{P^{-1} =}$

$$[T]^{B^{-1}} = \begin{bmatrix} -3/11 & -56/11 \\ -2/11 & 3/11 \end{bmatrix}$$