Eiganvalues and Eiganvectors.

* Task : 1 : Eiganvalue and Eiganvector.

1 $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Let,

Chareteastic equation,

$$\det(A - \lambda I) = 0$$

Or

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad — \quad (1)$$

where,

$$S_1 = 11$$

$$S_2 = (15-1) + (9-1) + (15-1)$$
$$= 14 + 8 + 14$$
$$= 36$$

$$S_3 = 3(15-1) + 1(-3+1) + 1(1-5)$$
$$= 3(14) + 1(-2) + 1(-4)$$

$$S_3 = 42 - 2 - 4$$

$$S_3 = 36$$

By eq^n 1,

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$
$$= \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$$
$$\therefore \lambda^2(\lambda - 2) - 9\lambda(\lambda - 2) + 18(\lambda - 2) = 0$$

$$\therefore (\lambda - 2)(\lambda^2 - 9\lambda + 18) = 0$$

$$\therefore (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$$\therefore \lambda = 2, 3, 6$$

=> Eigen Vector for $\lambda = 2$

$$\therefore [A - \lambda I][x] = 0$$

$$\therefore \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad — \ 2$$

-> For $(\lambda = 2)$ put in eq^n 2,

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$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 - x_1 + x_1 = 0$

$\therefore -x_2 + 3x_2 - x_2 = 0$

$\therefore x_3 - x_3 + x_3 = 0$

-> By Crammer's Rule,

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{1-3} = \frac{-x_2}{-1+1} = \frac{x_3}{3-1} = t$$

$$\therefore \frac{x_1}{-2} = \frac{-x_2}{0} = \frac{x_3}{2} = t$$

$\therefore x_1 = -2t$, $x_2 = 0t$

$x_3 = 2t$

-> $Sol^n = X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ 2t \end{bmatrix} = 2t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 2$ Eigen Vector = $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

=> Eigen Vector for $\lambda = 3$

By eq^n 2,

$$\therefore \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_1 + x_1 = 0$$
$$\therefore -x_2 + 2x_2 - 1x_2 = 0$$
$$\therefore x_3 - x_3 = 0$$

-> By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{(-1)} = \frac{-x_2}{1} = \frac{x_3}{-2+1} = t$$

$$\therefore x_1 = -t, \quad x_2 = -t$$

$$x_3 = -t$$

$\Rightarrow$ Sol^n = $t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 3$, Eigen Vector = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Eigen Vector for $\lambda = 6$

By eq^n 2,

$$\therefore \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 - x_1 + x_1 = 0$

$\therefore -x_2 - x_2 - x_2 = 0$

$\therefore x_3 - x_3 - 3x_3 = 0$

$\Rightarrow$ By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{(3-1)} = \frac{-x_2}{(-1-3)} = \frac{x_3}{(1+1)} = t$$

$\therefore x_1 = 2t, \quad x_2 = 4t, \quad x_3 = 2t$

$\rightarrow$ $Sol^n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

For $\lambda = 6$ Eigen vector $= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

2 $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Let,

charateastic equation

$\det(A - \lambda I) = 0$

or

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ — 1

Where,

$S_1 = 4$

$S_2 = (-9+8) + (-12+6) + (12-6$

$= -1 - 6 + 6$

$S_2 = -1$

$S_3 = 4(-9+8) - 6(-3+2)$
$+ 6(-4+3)$

$S_3 = 4(-1) - 6(-1) + 6(-1)$
$= -4 + 6 - 6$
$= -4$

Ex eq^n 1.

$\therefore \lambda^3 - \lambda^2$

$\therefore \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$
$\therefore \lambda^3 - \lambda^2 - 2\lambda^2 + 2\lambda - \lambda + 4 = 0$
$\therefore \lambda^2(\lambda - 1) - 2\lambda(\lambda - 1) - \lambda + 4 = 0$

$\therefore \lambda = 1 , \lambda^2 - 2\lambda =$

$\therefore \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$
$\therefore \lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda - 4\lambda + 4 = 0$
$\therefore \lambda^2(\lambda - 1) - 3\lambda(\lambda - 1) - 4(\lambda - 1) = 0$

$\therefore (\lambda - 1)(\lambda - 4)(\lambda + 1) = 0$

$\therefore \lambda = 1, 4, -1$

$$\rightarrow \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad — (1)$$

=> Eigen Vector For $\lambda = 1$

By eq^n 1,

$$\therefore \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 3x_1 + 6x_1 + 6x_1 = 0$$

$$\therefore x_1 + 2x_1 + 2x_1 = 0$$

$$\therefore x_2 + 2x_2 + 2x_2 = 0 \quad -$$

$$\therefore -x_3 - 4x_3 - 2x_3 = 0 \quad -$$

-> By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} 2 & 2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{(-4+8)} = \frac{-x_2}{(+2-2)} = \frac{x_3}{(-4+2)} = t$$

$$\therefore x_1 = 2t, \quad x_2 = 0t, \quad x_3 = -2t$$

-> $Sol^n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ 4t \\ -2t \end{bmatrix} = 2t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

→ For $\lambda = 1$, Eigen Vector : $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

⇒ Eigen Vector For $\lambda = -1$

By eq^n 1,

$$\therefore \begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ 1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 5x_1 + 6x_1 + 6x_1 = 0$ —

$\therefore \quad x_2 + 4x_2 + 2x_2 = 0$ —

$\therefore \quad x_3 - 4x_3 - 2x_3 = 0$

→ By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{12-24} = \frac{-x_2}{6-10} = \frac{x_3}{20-6} = t$$

$\therefore \quad x_1 = -12t$, $x_2 = -4t$

$x_3 = 14t$

$\rightarrow$ Sol^n = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12t \\ -4t \\ 14t \end{bmatrix} = -2t \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$

$\Rightarrow$ Eigen Vector for $\lambda = 4$

By eq^n 1,

$$\therefore \begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 6x_1 + 6x_1 = 0$

$\therefore x_2 - x_2 + 2x_2 = 0$

$\rightarrow$ By Crammer rule,

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & 0 \\ 2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 6 \\ -1 & -1 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{(12+6)} = \frac{-x_2}{(-6)} = \frac{x_3}{6} = t$$

$\therefore x_1 = 18t, \quad x_2 = 6t, \quad x_3 = 6t$

$\rightarrow$ Sol^n = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18t \\ 6t \\ 6t \end{bmatrix} = 6t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

3 $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Let,

chareteastic equation,

$\det(A - \lambda I) = 0$

or

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ — ①

Where,

$S_1 = 18$

$S_2 = (21-16) + (24-4) + (56-36)$
$= 5 + 20 + 20$
$= 45$

$S_3 = 8(21-16) + 6(-18+8)$
$+ 2(24-14)$

$S_3 = 40 - 60 + 20$

$S_3 = 0$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$
$\therefore \lambda^3 - 2\lambda^2 - 16\lambda^2$
$\therefore \lambda^2(\lambda - 2)$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\therefore \lambda^3 - 15\lambda^2 - 3\lambda^2 + 45\lambda = 0$

$\therefore \lambda^2(\lambda - 15) - 3\lambda(\lambda - 15) = 0$

$\therefore (\lambda^2 - 3\lambda)(\lambda - 15) = 0$

$\therefore \lambda = 3, 15$

$\rightarrow$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad — (2)$$

$\Rightarrow$ For $\lambda = 3$, Eigen vector

By eq^n 2,

$$\therefore \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 5x_1 - 6x_1 + 2x_1 = 0$

$\therefore -6x_2 + 4x_2 - 4x_2 = 0$

$\rightarrow$ By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}} = t$$

$\therefore \dfrac{x_1}{(24-8)} = \dfrac{-x_2}{(-12+20)} = \dfrac{x_3}{(20-36)} = t$

$\therefore x_1 = 16t, \quad x_2 = -8t, \quad x_3 = -16t$

$\Rightarrow \text{Sol}^n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ -8t \\ -16t \end{bmatrix} = -8t \begin{bmatrix} -2 \\ 1 \\ +2 \end{bmatrix}$

$\Rightarrow$ For $\lambda = 15$ Eigen Vector.

By eq^n 2,

$\therefore \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -7x_1 - 6x_1 + 2x_1 = 0$

$\therefore -6x_2 - 8x_2 - 4x_2 = 0$

$\rightarrow$ By Crammer rule,

$\therefore \dfrac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}} = t$

$\therefore \dfrac{x_1}{(24+16)} = \dfrac{-x_2}{(-12-28)} = \dfrac{x_3}{(56-36)} = t$

$\therefore x_1 = 40t, \quad x_2 = \overset{40}{10}t, \quad x_3 = 20t$

$\Rightarrow$ $Sol^n = \begin{bmatrix} 40t \\ 16t \\ 20t \end{bmatrix} = 4t \begin{bmatrix} 10 \\ \cancel{14}0 \\ 54 \end{bmatrix}$

$\Rightarrow$ Eigen vector for $\lambda = 15$ is $\begin{bmatrix} 10 \\ 10 \\ 54 \end{bmatrix}$

4 $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Let,

chareteastic equation,

$\det(A - \lambda I) = 0$

or

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where, $S_1 = 0$

$S_2 = -1 + (-1) + (-1)$
$= -3$

$S_3 = 2$

$\therefore \lambda^3 - 3\lambda = 0$

$\therefore \lambda^2 = 3\lambda$

$\therefore \lambda^2 = 3$

$\therefore \cancel{\lambda = \sqrt{3}}$

$\therefore \lambda^3 - 3\lambda - 2 = 0$

$\therefore \lambda^3 + \lambda^2 - \lambda^2 - 2\lambda - \lambda - 2 = 0$

$\therefore \lambda^2(\lambda+1) - \lambda(\lambda+1) - 2(\lambda+1) = 0$

$\therefore (\lambda+1)(\lambda^2-2)(\lambda+1) = 0$

$\therefore \lambda = -1, 2, 1$

=>

$[A - \lambda I][X] = 0$

$$\therefore \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad — (1)$$

=> For $\lambda = -1$

By eq^n 1,

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore x_1 + x_2 + x_3 = 0$$

Suppose $x_1 = t_1$, $x_2 = t_2$
$x_3 = (-t_1 + t_2)$

$\Rightarrow$ Eigen Vector For $\lambda = -1$

$$X = \begin{bmatrix} t_1 \\ t_2 \\ -t_1 + t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\Rightarrow$ For $\lambda = 2$

By $eq^n - 1$

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore -2x_1 + x_2 + x_3 = 0$$
$$\therefore x_1 - 2x_2 + x_3 = 0$$

By Crammer rule,

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = t$$

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$\therefore \frac{x_1}{3} = \frac{-x_2}{-3} = \frac{x_3}{3} = t$

$\Rightarrow X = \begin{bmatrix} 3t \\ 3t \\ 3t \end{bmatrix}$

$= 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ For $\lambda = 2 \Rightarrow$ Eigen Vector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

5 $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Let,
chareteastic equation,

$$\det(A - \lambda I) = 0$$

or

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

where, $S_1 = 5$

$$S_2 = (4-2) + (2+2) + (2)$$
$$= 2 + 4 + 2$$
$$= 8$$

$$S_3 = 1(4-2) - 2(1) + 2(2)$$

$$= 2 - 2 + 4$$
$$= 4$$

$\therefore \lambda^3 - 5\lambda^2 + 8\lambda + 4 = 0$

$\therefore \lambda^2(\lambda - 1) - 4\lambda(\lambda - 1) + 4(\lambda - 1) = 0$

$\therefore (\lambda - 2)^2 (\lambda - 1) = 0$

$\therefore \lambda = 2, 2, 1$

$\rightarrow$ $[A - \lambda I][x] = 0$

$$\therefore \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{--- (1)}$$

$\Rightarrow$ For $\lambda = 2$, By eq^n 1,

$$\therefore \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore -x_1 + 2x_2 + 2x_3 = 0$$
$$x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{-1} = \frac{x_3}{0} = t$$

$$\therefore x_1 = 2t, \quad x_2 = t$$

$$\rightarrow X = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ For $\lambda_2 = 1$ By eq^n - 1

$$\therefore \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore 2x_2 + 2x_3 = 0$

$\therefore -x_1 + 2x_2 + x_3 = 0$

By Crammer rule,

$$\therefore \frac{x_1}{\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{-2} = \frac{-x_2}{2} = \frac{x_3}{2} = t$$

$\therefore x_1 = -2t, \quad x_2 = -t, \quad x_3 = 2t$

$$\rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

6 $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

By char. equation,

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 3 & 3-\lambda \end{vmatrix} = 0 \quad \text{— (1)}$$

or

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where, $S_1 = 3$

$S_2 = (+3) = 3$

$S_3 = -1(-1) = 1$

$\therefore \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$\therefore (\lambda - 1)^3 = 0$

$\therefore \lambda = 1, 1, 1$

By eq^n, 1

$$\therefore \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -x_1 + x_2 = 0$

$\therefore -x_2 + x_3 = 0$

By Crammer Rule,

$$\therefore \frac{x_1}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = t$$

$$\therefore \frac{x_1}{1} = \frac{-x_2}{-1} = \frac{x_3}{1} = t$$

$$\therefore x_1 = t, \quad x_2 = t, \quad x_3 = t$$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ For $\lambda = 1 \Rightarrow$ Eigen vector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Unit : 5 Eigen values and Eigen Vector.

* Task : 2 Algebraic and Geometric of an Eigenvalue.

1 $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

By char. equation,

$$\det(A - \lambda I) = 0$$

or

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where,

$$S_1 = 6$$

$$S_2 = (12) + (-3-6) + (-4+12)$$
$$= 11$$

$$S_3 = -1(12) - 4(-9) + (-2)(-3+12)$$
$$= -12 + 36 - 18 = 6$$

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$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore \quad \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore \quad \lambda^2(\lambda - 1) - 5\lambda(\lambda - 1) + 6(\lambda - 1) = 0$

$\therefore \quad (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\therefore \quad (\lambda - 1)(\lambda - 3)(\lambda - 2) = 0$

$\therefore \quad \lambda = 1, 3, 2$

$\Rightarrow$ For $\lambda = 1$

Algebriac Multiplication is 1.

Here,

$[A - \lambda I][x] = 0$

$$\therefore \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} = 0 \quad \text{— (1)}$$

$$\therefore \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} = 0$$

$$R_1 \rightarrow R_1/-2$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 \rightarrow R_2 / -3$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 \rightarrow R_2 - R_1$$
$$R_3 \rightarrow R_3 + 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 / -5$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, Rank of matrix is 2

$\rightarrow$ Geometric Multiplication $= \cancel{Rank - 3}$ $3 - Rank$

$= 3 - 2$

$= 1$

$\Rightarrow$ For $\lambda = 3$

By eq^n, 1

$$\therefore \left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$\therefore \left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$$

$$R_2 \to R_2 + 3R_1$$
$$R_3 \to R_3 + 3R_1$$

$$\therefore \left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ 0 & 10 & -6 & 0 \\ 0 & 10 & -6 & 0 \end{array}\right]$$

$$R_2 \to R_2/10 \quad , \quad R_3 \to R_3/10$$

$$\therefore \left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 1 & -3/5 & 0 \end{array}\right]$$

$$\cancel{R_3 \to R_3 - 10R}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ G.M. $= 3 - 2 = 1$

$\Rightarrow$ For $\lambda = 2$

By eq^n 1,

$$\therefore \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 / -3$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 \rightarrow R_2 + 3R_1$$
$$R_3 \rightarrow R_3 + 3R_1$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 / -2$, $\quad R_3 \rightarrow R_3 / -3$

$$\therefore \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\therefore \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\to$ G.M. $= 3 - 2 = 1$

2 $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

By char. equation,

$\det(A - \lambda I) = 0$

or

$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where $S_1 = 1$

$S_2 = (21 - 32) + (-63 + 64) + (-27 + 32)$
$= -5$

$$S_3 = -9(21-32) - 4(-56+64) + 4(-64+48)$$

$$= +9(11) - 4(8) + 4(-16)$$
$$= 3$$

$\Rightarrow \quad \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\therefore \quad \lambda^3 + \lambda^2 - 2\lambda^2 - 2\lambda - 3\lambda - 3 = 0$

$\therefore \quad \lambda^2(\lambda+1) - 2\lambda(\lambda+1) - 3(\lambda+1) = 0$

$\therefore \quad (\lambda+1)(\lambda^2 - 2\lambda - 3) = 0$

$\therefore \quad (\lambda+1)(\lambda+1)(\lambda-3) = 0$

$\therefore \quad \lambda = -1, -1, 3$

$\Rightarrow$ For $\lambda = -1$

Algebriac Multiplication is 2.

$\therefore \quad [A - \lambda I][x] = 0$

$$\therefore \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad — (1)$$

By eq^n 1,

$$\therefore \left[\begin{array}{ccc|c} -8 & 4 & 4 & 0 \\ -8 & 4 & 4 & 0 \\ -16 & 8 & 8 & 0 \end{array}\right] = 0$$

$$R_1 \to R_1/-8, \quad R_2 \to R_2/-8, \quad R_3 \to R_3/8$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & -1/2 & -1/2 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 + 2R_1$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of matrix is 1.

$\to$ G.M. $= 3 - 1 = 2$

$\Rightarrow$ For $\lambda = 3$

Algebriac Multiplication is 1.

By eq^n 1,

$$\therefore \left[\begin{array}{ccc|c} -8 & 4 & 4 & 0 \\ -8 & 0 & 4 & 0 \\ -16 & 8 & 4 & 0 \end{array}\right]$$

$$R_1 \to R_1/-8, \quad R_2 \to R_2/-8, \quad R_3 \to R_3/8$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & 0 & -1/2 & 0 \\ -2 & 1 & 1/2 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 2R_1$

$$\therefore \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of matrix is 2.

$\to$ G.M. $= 3 - 2 = 1$

3 $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

By char. equation,

$$\det(A - \lambda I) = 0$$

or

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Where,

$S_1 = 5$

$S_2 = 2 + 2 + 4 = 8$

$$S_3 = 2(2) - 3(0) + 4(0)$$
$$= 4$$

$\therefore \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\therefore \lambda^3 - \lambda^2 - 4\lambda^2 + 4\lambda + 4\lambda - 4 = 0$

$\therefore \lambda^2(\lambda - 1) - 4\lambda(\lambda - 1) + 4(\lambda - 1) = 0$

$\therefore (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

$\therefore (\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\therefore \lambda = 1, 2, 2$

=> For $\lambda = 1$

Algebric Multiplication is 1.

$\therefore [A - \lambda I][x] = 0$

$$\therefore \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{--- (1)}$$

By eq^n 1,

$$\therefore \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of matrix is 2.

$\Rightarrow$ G.M. $= 3 - 2 = 1$

$\Rightarrow$ For $\lambda = 2$

Algebric Multiplication is 2.

By eq^n 1,

$$\therefore \left[\begin{array}{ccc|c} 0 & 3 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\therefore \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Rank of matrix is 2.

$\Rightarrow$ G.M $= 3 - 2 = 1$

4 $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

By char. equation,

$\det(A - \lambda I) = 0$

or

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where, $S_1 = 3$

$$S_2 = -3 + (0) + 0$$
$$= -3$$

$$S_3 = 0(-3) - 1(0) + 0$$
$$= 0$$

$$\therefore \lambda^3 + 3\lambda^2 + 3\lambda = 0$$

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* Task : 4 : Cayley - Hamilton theorem.

(ii) Verify Cayley-Hamilton theorem for the matrix and Find A^4 and A^{-1}.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

By char. equation,

$$\det(A - \lambda I) = 0$$

or

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

where, $S_1 = 6$

$$S_2 = (4-1) + (4-1) + (4-1)$$
$$= 3 + 3 + 3$$
$$= 9$$

$$S_3 = 2(3) + 1(-2+1) + 1(1-2)$$
$$= 6 - 1 - 1$$
$$= 4$$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To prove, $A^3 - 6A^2 + 9A - 4I = 0$ — (1)

$$\Rightarrow A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -2-2-1 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\Rightarrow A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

put A^3 and A^2 and A value in eq^n 1,

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$$L.H.S. = A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & 9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L.H.S. = R.H.S.$$

Hence, Cayley-Hamilton theorem is Verify.

$\Rightarrow$ For A^4

multiply A to eq^n 1,

$$\therefore A^4 - 6A^3 + 9A^2 - 4A = 0$$

$$\therefore A^4 = 6A^3 - 9A^2 + 4A$$

$$\therefore A^4 = 6\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 9\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 4\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

(i) Verify Cayley - Hamilton theorem for the matrix and Find A^{-1}.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

by char, equation,

$$\det(A - \lambda I) = 0$$

or

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where,

$$S_1 = 4$$

$$S_2 = (2-6) + (1-7) + (2-12)$$
$$= -4 - 6 - 10$$
$$= -20$$

$$S_3 = 1(-4) - 3(4-3) + 7(8-2)$$
$$= -4 - 3 + 42$$
$$= 35$$

$$\therefore \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

To prove,

$$A^3 - 4A^2 - 20A - 35 = 0$$

$$\rightarrow A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$