Unit : 6 Vector Differential Calculus.

* Task - 1 : Velocity and Acceleration, Tangent Vector and Arc Length of a curve.

1 Find the velocity and aceleration of a particle which move along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. at given time $t > 0$. also find the magnitude of velocity and acceleration.

Here, given curve $x = 2\sin 3t$
$y = 2\cos 3t$
$z = 8t$

$$\vec{r}(t) = 2\sin 3t\,\hat{i} + 2\cos 3t\,\hat{j} + 8t\,\hat{k}$$

→ Velocity $\dfrac{d\vec{r}}{dt} = 6\cos 3t\,\hat{i} - 6\sin 3t\,\hat{j} + 8\,\hat{k}$

→ acceleration $\bar{a} = \dfrac{d^2\vec{r}}{dt} = -18\sin 3t\,\hat{i} - 18\cos 3t\,\hat{j}$

$\rightarrow \|\vec{v}\| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$

$= \sqrt{36(\cos^2 3t + \sin^2 3t) + 64}$

$= \sqrt{36 + 64}$

$\|\vec{v}\| = 10$

$\rightarrow \|\vec{a}\| = \sqrt{18^2 \sin^2 3t + 18^2 \cos^2 3t + (1)^2}$

$= \sqrt{18^2(\sin^2 3t + \cos^2 3t)}$

$\|\vec{a}\| = 18$

2. Find the length of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ in the first quadrant.

Here, given $x = a\cos^3\theta$, $y = a\sin^3\theta$
$\theta \rightarrow 0$ to $\pi/2$

$d\ \vec{r}(\theta) = a\cos^3\theta\ \hat{i} + a\sin^3\theta\ \hat{j}$

$\frac{d\vec{r}(\theta)}{d\theta} = -3a\cos^2\theta\sin\theta\ \hat{i} + 3a\sin^2\theta\cos\theta\ \hat{j}$

$= 3a\sin\theta\cos\theta(\sin\theta\ \hat{j} - \cos\theta\ \hat{i})$

$\left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}$

$$= 3a\sin\theta\cos\theta$$

→ Length of curve $= \int_0^{\pi/2} \left|\frac{dr}{d\theta}\right| \cdot d\theta$

$$= \int_0^{\pi/2} \frac{3a}{2}\sin 2\theta \cdot d\theta$$

$$= \frac{3a}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{3a}{4}\left[-1-1\right]$$

$$= -\frac{3a}{4}(-2)$$

$$= \frac{3a}{2}$$

E 3 A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$, where t represent time. Find the components of its velocity and accelaration at $t=1$ in the direction $2\hat{i}+3\hat{j}+6\hat{k}$

Here, given $x = t^3 + 1$, $y = t^2$
and $z = 2t + 5$
$t = 1$

-> $\vec{r}(t) = t^3 + 1\,\hat{i} + t^2\hat{j} + 2t + 5\,\hat{k}$

-> Velocity $\vec{v}$

$$\frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\vec{v}_{(t=1)} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

-> accelaration $\vec{a}$

$$\frac{d^2\vec{r}}{dt} = 6t\hat{i} + 2\hat{j}$$

$$\vec{a}_{(t=1)} = 6\hat{i} + 2\hat{j}$$

-> direction $\bar{n} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$|\vec{n}| = \sqrt{4 + 9 + 36} = 7$$

$$\hat{n} = \frac{\bar{n}}{|\vec{n}|} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

→ Velocity at direction $\bar{n}$

$$\bar{v} \cdot \hat{n} = (3\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}$$

$$\bar{v} \cdot \hat{n} = \frac{6 + 6 + 12}{7}$$

$$= \frac{24}{7}$$

→ accelaration at direction $\bar{n}$

$$\bar{a} \cdot \hat{n} = (6\hat{i} + 2\hat{j}) \cdot \frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}$$

$$= \frac{12 + 6}{7}$$

$$= \frac{18}{7}$$

4 Find the general formula for the tangent vector and unit tangent vector to the curve given by $\bar{r}(t) = t^2\hat{i} + 2\sin t\,\hat{j} + 2\cos t\,\hat{k}$

Here, Given $\bar{r}(t) = t^2\hat{i} + 2\sin t\,\hat{j} + 2\cos t\,\hat{k}$

→ Tangent Vector $\bar{T}$

$$\bar{T} = \frac{d\bar{r}}{dt} = 2t\,\hat{i} + 2\cos t\,\hat{j} - 2\sin t\,\hat{k}$$

$$|\bar{T}| = \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{4t^2 + 4(\cos^2 t + \sin^2 t)}$$

$$= \sqrt{4t^2 + 4}$$

$$= 2\sqrt{t^2 + 1}$$

→ Unit Tangent Vector

$$\hat{T} = \frac{\bar{T}}{|\bar{T}|} = \frac{2t\,\hat{i} + 2\cos t\,\hat{j} - 2\sin t\,\hat{k}}{2\sqrt{t^2+1}}$$

$$= \frac{t\,\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}}{\sqrt{t^2+1}}$$

5 Find the unit tangent vector $\bar{T}(t)$ for $(2t, 3t^2, 4t^3) = \bar{r}(t)$ at $t = 1$.

Here, given $\bar{r}(t) = 2t\hat{i} + 3t^2\hat{j} + 4t^3\hat{k}$

→ Tangent Vector

$\bar{T} = 2\hat{i} + 6t\hat{j} + 12t^2\hat{k}$

$\bar{T}_{(t=1)} = 2\hat{i} + 6\hat{j} + 12\hat{k}$

$|\bar{T}| = \sqrt{4+36+144}$

$= 2\sqrt{46}$

→ Unit Tangent Vector

$\hat{T} = \dfrac{\bar{T}}{|\bar{T}|} = \dfrac{2\hat{i} + 6\hat{j} + 12\hat{k}}{2\sqrt{46}}$

$= \dfrac{\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{46}}$

(vi)

5 Find the length of curve $\bar{r}(t) = \frac{2\sqrt{2}}{3}t^{3/2}\hat{i} + \frac{t^2}{2}\hat{j} + (t+3)\hat{k}$

between $t=0$ and $t=2$

Here, Given $\vec{r}(t) = \frac{2\sqrt{2}}{3} t^{3/2}\hat{i} + \frac{t^2}{2}\hat{j} + (t+3)\hat{k}$

$\rightarrow \frac{d\bar{r}}{dt} = \frac{3}{2} \cdot t^{1/2} \cdot \frac{2\sqrt{2}}{3}\hat{i} + \frac{2t}{2}\hat{j} + \hat{k}$

$\frac{d\bar{r}}{dt} = \sqrt{2}\, t^{1/2}\hat{i} + t\hat{j} + \hat{k}$

$\rightarrow \left|\frac{d\bar{r}}{dt}\right| = \sqrt{2t + t^2 + 1}$

$= \sqrt{(t+1)^2}$

$= t + 1$

$\rightarrow$ length of curve $= \int_0^2 t + 1 \cdot dt$

$= \left[\frac{t^2}{2} + t\right]_0^2$

$= \frac{4}{2} + 2$

$= 4$

* Task: 2 Gradient of a scalar field and Directional Derivative.

1 If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that,

(a) $\nabla \log r = \frac{\hat{r}}{r}$

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \quad \text{— (1)}$$

$\rightarrow$ $\nabla \log r = \nabla \log(\sqrt{x^2+y^2+z^2})$

$$= \frac{2x\,\hat{i}}{2\sqrt{x^2+y^2+z^2}} + \frac{2y\,\hat{j}}{2\sqrt{x^2+y^2+z^2}} + \frac{2z\,\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

By eq - 1

$\nabla \log r = \hat{r}$

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(b) $\nabla(r^n) = n r^{n-2} \cdot \bar{r}$

$\rightarrow$ L.H.S. $= \nabla r^n$

$= \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n$

$= \nabla (x^2+y^2+z^2)^{n/2}$

$\cancel{= \nabla}$

$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (2x\hat{i}+2y\hat{j}+2z\hat{k})$

$= n(x^2+y^2+z^2)^{n-2} \cdot (x\hat{i}+y\hat{j}+z\hat{k})$

Here, we know that,

$r = x^2+y^2+z^2$ and
$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$= n \cdot r^{n-2} \cdot \bar{r}$

L.H.S. = R.H.S.

2 Find the gradient of

$$F(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1}(xz)$$
at $P(1,1,1)$

$\rightarrow$ $$\nabla F(x,y,z) = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

$$= \left(-6xz + \frac{1 \cdot z}{1+x^2z^2}\right)\hat{i} +$$

$$-6yz\,\hat{j} + \left(6z^2 + \frac{x}{1+x^2z^2} - 3(x^2+y^2)\right)\hat{k}$$

$\rightarrow$ $$\nabla F(x,y,z)_{(1,1,1)} = -6 + \frac{1}{2}\hat{i} + -6\hat{j} + 6 + \frac{1}{2}\hat{k} - 6\hat{k}$$

$$= \cancel{-5\hat{i} - 3}$$

$$= -\frac{11}{2}\hat{i} - 6\hat{j} + \frac{13}{2}\hat{k} - 6\hat{k}$$

$$= -\frac{11}{2}\hat{i} - 6\hat{j} + \frac{1}{2}\hat{k}$$

3 Find the unit normal vector of the cone of the revolution $z^2 = 4(x^2+y^2)$ at the point $P(1,0,2)$

Here, given $f_{(x,y,z)} = 4x^2+4y^2-z^2$

$\rightarrow \nabla f_{(x,y,z)} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$

$\rightarrow \nabla f_{(1,0,2)} = 8\hat{i} - 4\hat{k}$

$\rightarrow$ Unit normal vector

$\hat{n} = \frac{8\hat{i} - 4\hat{k}}{\sqrt{64+16}}$

$= \frac{8\hat{i} - 4\hat{k}}{4\sqrt{15}}$

$= \frac{2}{\sqrt{15}}\hat{i} - \frac{1}{\sqrt{15}}\hat{k}$

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4 Q Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

(a) increses most rapidly
(b) decreses most rapidly at point (1,1).

Here, given $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

$\rightarrow \nabla f = \frac{2x}{2}\hat{i} + \frac{2y}{2}\hat{j}$

$= x\hat{i} + y\hat{j}$

$\rightarrow \nabla f_{(1,1)} = \hat{i} + \hat{j}$

$\rightarrow$ Direction $= \frac{\hat{i}+\hat{j}}{\sqrt{1+1}} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

a $\nabla f_{(1,1)} = \hat{i} + \hat{j}$

b $-\nabla f_{(1,1)} = -\hat{i} - \hat{j}$

5 Find the directional derivative of $\phi(x, y, z) = 4xz^3 - 3x^2y^2z$ at point $P(2, -1, 2)$

(a) in the direction $2\hat{i} + 3\hat{j} + 6\hat{k}$

(b) along the direction normal to the surface $x^2+y^2+z^2 = 9$ at $(1, 2, 3)$.

→ Here, given, $\phi_{(x, y, z)} = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (4z^3 - 6xy^2z\,\hat{i}) + (-6x^2yz\,\hat{j})$$

$$+ (12xz^2 - 3x^2y^2\,\hat{k})$$

→ $$\nabla\phi_{(2,-1,2)} = (4(2)^3 - 6(2)(-1)^2(2))\hat{i}$$

$$- 6(2)^2(-1)(2)\,\hat{j}$$

$$+ (12(2)(2)^2 - 3(2)^2(-1)^2)\hat{k}$$

$$\nabla\phi_{(2,-1,2)} = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

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a

$|\bar{a}| = \sqrt{4+9+36}$

$= 7$

$\hat{a} = \frac{\bar{a}}{|a|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$

→ Directional Derivatives $= \nabla\phi \cdot \hat{a}$

$= (8\hat{i}+48\hat{j}+84\hat{k}) \cdot \frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$

$= \frac{664}{7}$

b Normal surface

$\phi_1 = x^2+y^2+z^2-9$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_{1\,(1,2,2)} = 2\hat{i}+4\hat{j}+4\hat{k}$

→ Directinal Derivaties $= \nabla\phi \cdot \frac{\nabla\phi_1}{|\nabla\phi_1|}$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k} \cdot \frac{(2\hat{i} + 4\hat{j} + 6\hat{k})}{\sqrt{4 + 16 + \cancel{36}}}$$

16

$$= \frac{16 + 142 + 336}{6}$$

$$= \frac{272}{3}$$

6 The temperature at any ponit in space is given by $T = xy + yz + zx$. Determine of T in the direction $3\hat{i} - 4\hat{k}$ at the point $P(1,1,1)$

→ Here given $F_{(x,y,z)} = xy + yz + zx$

direction $\bar{a} = 3\hat{i} - 4\hat{k}$

$$|\bar{a}| = \sqrt{9 + 16} = 5$$

→ $$\nabla F_{(x,y,z)} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \underset{+z}{y}\hat{i} + \underset{+x}{z}\hat{j} + \underset{+y}{x}\hat{k}$$

$$\nabla f_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

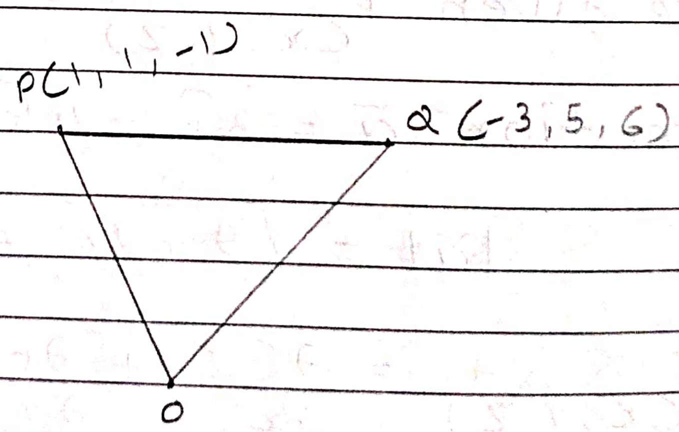
→ Directinal Derivatives $= \nabla f_{(1,1,1)} \cdot \dfrac{\bar{a}}{|\bar{a}|}$

$$= 2\hat{i} + 2\hat{j} + 2\hat{k} \cdot \frac{(3\hat{i} - 4\hat{k})}{5}$$

$$= \frac{6 - 8}{5}$$

$$= \frac{-2}{5}$$

7 Find the directinal Derivatives of $\phi = 4e^{2x-y+z}$ at point $P(1,1,-1)$ in the direction towarbs the point $Q(-3,5,6)$.



Here given, $\phi_{(x,y,z)} = 4e^{2x - y + z}$

$\vec{a} = (-3-1)\hat{i} + (5-1)\hat{j} + (6+1)\hat{k}$

$= -4\hat{i} + 4\hat{j} + 7\hat{k}$

$\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{\sqrt{16+16+49}} = \dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{9}$

$\rightarrow \nabla \phi_{(x,y,z)} = 8e^{2x-y+2}\hat{i} - 4e^{2x-y+2}\hat{j}$
$+ 4e^{2x-y+2}\hat{k}$

$\nabla \phi_{(1,1,-1)} = 8\hat{i} - 4\hat{j} + 4\hat{k}$

$\rightarrow$ Directinal Derivatives

$= \nabla \phi \cdot \hat{a}$

$= 8\hat{i} - 4\hat{j} + 4\hat{k} \cdot \left(\dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{9}\right)$

$= \dfrac{-32-16+28}{9}$

$= \dfrac{-20}{9}$

* Task : 3 Divergence, Curl

1 Find the divergence and curl where $\bar{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$.

Here given, $\bar{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

Let $F_1 = xz^3$, $F_2 = -2x^2yz$, $F_3 = 2yz^4$

$\rightarrow$ $\operatorname{div}(\bar{F}) = \vec{\nabla} \cdot \vec{F}$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= z^3 \cancel{+} - 2x^2z + 8yz^3$$

$\rightarrow$ $\operatorname{curl}(\bar{F}) = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} \quad \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$= (2z^4 + 2x^2y)\hat{i} + (3xz^2)\hat{j}$
$-4xyz\hat{k}$

$curl(\bar{F}) = (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j}$
$-4xyz\hat{k}$

2 If $\phi = x^3 + y^3 + z^3 - 3xyz$. Find
a) $\bar{r} \cdot \nabla\phi$ b) $div(\bar{F})$ c) $curl(\bar{F})$

-) Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

~~Where F_1~~
$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F} = \nabla\phi = 3x^2\hat{i} - 3yz\hat{i} + (3y^2 - 3xz)\hat{j}$
$+ (3z^2 - 3xy)\hat{k}$

Here, $F_1 = 3x^2 - 3yz$
$F_2 = 3y^2 - 3xz$
$F_3 = 3z^2 - 3xy$

a $\bar{r} \cdot \nabla\phi = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3x^2 - 3yz\hat{i} +$
$3y^2 - 3xz\hat{j} + 3z^2$
$-3xy\hat{k}$

$= 3x^3 - 3yxz + 3y^3 - 3yxz$
$+ 3z^3 - 3yxz$

$= 3(x^2 + y^2 + z^3 - 3xyz)$

$= 3\phi$

b $\operatorname{div}(F) = \bar{\nabla} \cdot \bar{F}$

$= \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

$= 6x + 6y + 6z$

c $\operatorname{Curl}(F) = \bar{\nabla}' \times F'$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= (-3x + 3x)\hat{i} + (-3y + 3y)\hat{j}$

$+ (-3z + 3z)\hat{k}$

$\operatorname{curl}(\bar{F}) = 0$

3 If $\bar{a}$ is constant vector and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(a) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z$

$\rightarrow \nabla(\bar{a}\cdot\bar{r}) = \frac{\partial(\bar{a}\cdot\bar{r})}{\partial x}\hat{i} + \frac{\partial(\bar{a}\cdot\bar{r})}{\partial y}\hat{j} + \frac{\partial(\bar{a}\cdot\bar{r})}{\partial z}\hat{k}$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

(b) $\text{Curl}(\bar{a}\times\bar{r}) = 2\bar{a}$

$$\bar{a}\times\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$= (a_2z - a_3y)\hat{i} - \hat{j}(a_1z - a_3x)$

$+ \hat{k}(a_1y - a_2x)$

Let $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

where, $F_1 = a_2z - a_3y$, $F_2 = (-a_1z + a_3x)$, $F_3 = (a_1y - a_2x)$

$\Rightarrow$ $Curl(\bar{a} \times \bar{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} -$$

$$\hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) +$$

$$\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= [a_1 + a_1]\hat{i} + [a_2 + a_2]\hat{j}$$

$$+ [a_3 + a_3]\hat{k}$$

$$= 2a_1\hat{i} + 2a_2\hat{j} + 2a_3\hat{k}$$

$Curl(\bar{a} \times \bar{r}) = 2\bar{a}$

(c) $div(\bar{a} \times \bar{r} \times \bar{a}) = 2a^2$

$\bar{a} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= (a_2 z - a_3 y)\hat{i} - \hat{j}(a_1 z - a_3 x)$$
$$+ \hat{k}(a_1 y - a_2 x)$$

$\rightarrow (\bar{a} \times \bar{F}) \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= (a_3^2 x - a_1 a_3 z - a_2 a_1 y + a_2^2 x)\hat{i}$

$- \hat{j}(a_2 a_3 z - a_3^2 y - a_1' y + a_1 a_2 x)$

$+ \hat{k}(a_2^2 z - a_3 a_2 y - a_1 a_3 x + a_1^2 z)$

Let $\bar{F}' = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

where, $F_1 = a_3^2 x - a_1 a_3 z - a_2 a_1 y + a_2^2 x$

$F_2 = a_2 a_3 z - a_3^2 y - a_1 y + a_1 a_2 x$

$F_3 = a_2^2 z - a_3 a_2 y - a_1 a_3 x + a_1^2 z$

$\rightarrow \operatorname{div}(\bar{a} \times \bar{F} \times \bar{a}) = \bar{\nabla} \cdot \bar{F}$

$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= a_3^2 + a_2^2 + a_3^2 + a_1^2$

$+ a_2^2 + a_1^2$

$= 2(a_3^2 + a_2^2 + a_1^2)$

$= 2a^2$

$(\because \|\bar{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2})$

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4 Prove that,

a $r^n \bar{r}$ is irrotational.

Here, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\|\bar{r}\| = \sqrt{x^2 + y^2 + z^2}$

$\rightarrow r^n \cdot \bar{r} = r^n x\hat{i} + r^n y\hat{j} + z \cdot r^n \hat{k}$

$\rightarrow$ We know that,

$$\text{curl}(r^n \cdot \bar{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \left(\frac{\partial r^n z}{\partial y} - \frac{\partial}{\partial z}(r^n y)\right)\hat{i}$$

$$+ \left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x} r^n z\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)\hat{k}$$

$$= \left(n r^{n-1} z \cdot \frac{\partial r}{\partial y} - y \cdot r^{n-1} \cdot \frac{\partial r}{\partial z}\right)\hat{i}$$

$$+ \hat{j}\left(x n^{n-1} \cdot \frac{\partial r}{\partial z} - z n r^{n-1} \cdot \frac{\partial r}{\partial x}\right)$$

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$$+\hat{k}\left(ynr^{n-1}\cdot\frac{\partial r}{\partial x}-xnr^{n-1}\cdot\frac{\partial r}{\partial y}\right)$$

$\rightarrow \dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$

$\rightarrow \dfrac{\partial r}{\partial y} = \dfrac{2y}{2\sqrt{x^2+y^2+z^2}} = \dfrac{y}{r}$

$\rightarrow \dfrac{\partial r}{\partial z} = \dfrac{2z}{2\sqrt{x^2+y^2+z^2}} = \dfrac{z}{r}$

$\rightarrow \operatorname{curl}(r^n \cdot \bar{r}) = \left(znr^{n-1}\cdot\dfrac{y}{r} - ynr^{n-1}\dfrac{z}{r}\right)\hat{i}$

$+\left(xnr^{n-1}\cdot\dfrac{z}{r} - xnr^{n-1}\cdot\dfrac{z}{r}\right)\hat{j}$

$+\left(ynr^{-1}\cdot\dfrac{x}{r} - ynr^{-1}\dfrac{x}{r}\right)\hat{k}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= 0$

(b) $\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0$

$\rightarrow \frac{\bar{r}}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$= \frac{x\,\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \frac{y\,\hat{j}}{(x^2+y^2+z^2)^{3/2}}$

$+ \frac{z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\rightarrow \operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = \bar{\nabla}\cdot\vec{F}$

$= \frac{\partial F_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}x(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3}$

$+ \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}y(x^2+y^2+z^2)^{1/2}\cdot 2y}{(x^2+y^2+z^2)^3}$

$+ \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}z(x^2+y^2+z^2)^{1/2}\cdot 2z}{(x^2+y^2+z^2)^3}$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0$$

5 Is $\bar{F} = \dfrac{\bar{a} \times \bar{r}}{r^n}$ a solenoidal vector?

→ For Solenoidal vector $\operatorname{div}(F) = 0$

Here, $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

→ $\bar{a} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= \hat{i}(a_2 z - a_3 y) - \hat{j}(a_1 z - a_3 x) + \hat{k}(a_1 y - a_2 x)$$

→ $r^{-n}(\bar{a} \times \bar{r}) = r^{-n}(a_2 z - a_3 y)\hat{i}$

$- r^{-n}(a_1 z - a_3 x)\hat{j}$

$+ r^{-n}(a_1 y - a_2 x)\hat{k}$

$\rightarrow$ $div(\bar{F}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= r^{-n}(0) + r^{-n}(0) + r^{-n}(0)$

$= 0$

So, the vector is Solenoidal vector.

6 A vector field is given by
$\bar{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$.
Show that the field is irrotational and also find the scalar potential.

Let, $\vec{F} = F_1\hat{i} + F_2\hat{j}$

where $F_1 = (x^2 + xy^2)$
$F_2 = (y^2 + x^2y)$

$\rightarrow$ $Curl(\bar{F}) = \bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} +$

$\hat{j}\left(\frac{-\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}\right) +$

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$$\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\text{Curl}(\bar{F}) = 0$

Hence, F is irrotational.

-) For Scalar Potential,

$$\frac{\partial \phi}{\partial x} = x^2 + xy^2, \quad \frac{\partial \phi}{\partial y} = y^2 + x^2 y$$

$$-) \int \partial \phi = \int x^2 + xy^2 \cdot \partial x + y^2 + x^2 y \cdot \partial y$$

$$\phi = \frac{x^3}{3} + \frac{2x^2y^2}{2} + \frac{y^3}{3}$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + x^2y^2$$