Vector Space

* Task : 1 Vector Space.

1

(a) $(x_1, x_2) + (Y_1, Y_2) = (x_1 + Y_1, x_2 + Y_2)$

and $k(x_1, x_2) = (|k| x_1, |k| x_2)$

=) Let $V = \{(x_1, x_2), (Y_1, Y_2) \in R\}$

$U = (x_1, x_2)$, $V = (Y_1, Y_2)$, $W = (Z_1, Z_2) \in R$
and k_1, k_2 are scaler.

1 For $U, V \in R$

$U + V = (x_1, x_2) + (Y_1, Y_2)$
$= (x_1 + Y_1, x_2 + Y_2) \in V$

$\because x_1 + Y_1 \in R$
$\because x_2 + Y_2 \in R$

2 $U + V = (x_1 + Y_1, x_2 + Y_2)$ — 1

$V + U = (Y_1, Y_2) + (x_1, x_2)$
$= (x_1 + Y_1, x_2 + Y_2)$ — 2

Here eq^n 1 = 2

$\therefore U+V = V+U$

3 $U+(V+W) = (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)]$

$= (x_1, x_2) + (y_1+z_1, y_2+z_2)$

$= (y_1+z_1+x_1, x_2+y_2+z_2)$ — 1

$W+(U+V) = (z_1, z_2) + [(x_1, x_2) + (y_1, y_2)]$

$= (z_1, z_2) + [(x_1+y_1, y_2+x_2)]$

$= (y_1+z_1+x_1, x_2+y_2+z_2)$ — 2

Here eq^n 1 = 2

$\therefore U+(V+W) = W+(U+V)$

4 Let $a \in V$. $a = (a_1, b_1)$

$\therefore U + a = U$

$\therefore (x_1, x_2) + a = (x_1, x_2)$

$\therefore (x_1, x_2) + (a_1, b_1) = (x_1, x_2)$

$\therefore (x_1+a_1, x_2+b_1) = (x_1, x_2)$

$\therefore x_1 + a_1 = x_1 \Rightarrow a_1 = 0$

$\therefore x_2 + b_1 = x_2 \Rightarrow b_1 = 0$

$(a_1, b_1) = (0, 0) =$ zero vetor

5 Let $K \in V$, $K = (M, N)$

$\therefore U + K = 0$

$\therefore (x_1, x_2) + (M, N) = (0, 0)$

$\therefore (x_1 + M_1 , x_2 + N) = (0, 0)$

$\therefore x_1 + M_1 = 0 \Rightarrow M_1 = -x_1$

$\therefore x_2 + N = 0 \Rightarrow N = -x_2$

$K = (M, N) = (-x_1, -x_2) \in R$

6 Scaler K_1,

$K_1 U = (Kx_1, Kx_2) \in R$.

7 $K_1 (U + V) = K_1 (x_1 + y_1 , x_2 + y_2)$

$= (K_1(x_1 + y_1) , K_1(x_2 + y_2))$ — (1)

$\Rightarrow K_1 U + K_1 V = K_1 (x_1, x_2) + K_1 (y_1, y_2)$

$= K_1 (x_1 + y_1 , x_2 + y_2)$

$= (K_1 (x_1 + y_1) , K_1 (x_2 + y_2))$ — (2)

Here, eq^n 1 = 2

$\therefore K_1(U+V) = K_1 U + K_1 V$

8 $U(K_1 + K_2) = (x_1, x_2)(K_1 + K_2)$

$= K_1(x_1, x_2) + K_2(x_1, x_2)$

$= (K_1 x_1, K_1 x_2) + (K_2 x_1, K_2 x_2)$ — (1)

$\Rightarrow K_1 U + K_2 U = K_1(x_1, x_2) + K_2(x_1, x_2)$

$= (K_1 x_1, K_1 x_2) + (K_2 x_1, K_2 x_2)$ — (2)

Here, eq^n $1 \neq 2$

$\therefore U(K_1 + K_2) \neq K_1 U + K_2 U$

9 $K_1(K_2 U) = K_1(K_2 x_1, K_2 x_2)$

$= (K_1 K_2 x_1, K_1 K_2 x_2)$

$= K_1 K_2 (x_1, x_2)$

$= K_1 K_2 (U)$

10 Here, R^2 in not a vector space under the given operation.

6 $(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$

and $k(x_1, x_2) = (0, kx_2)$

$\Rightarrow V = \{(x, y) / x, y \in R\}$

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$, $w = (x_3, y_3)$ are vector in R^2

For $k_1, k_2 \in R$

1 $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in R$

For $u, v \in R$

2 $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$= (x_2, y_2) + (x_1, y_1)$

$u + v = v + u$

3 $u + (v + w) = u + [(x_2, y_2) + (x_3, y_3)]$

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$ — 1

$\rightarrow (U+V)+W = (x_2+x_2 , y_2+y_1) + (x_3, y_3)$

$= (x_1+x_2+x_3 , \; y_1+y_2+x_3)$

$\therefore U+(V+W) = (U+V)+W.$

4 Let $e = (e_1, e_2) \in R$

such that $U+e = U$

$\therefore (x_1, y_1) + (e_1, e_2) = (x_1, y_1)$

$\therefore x_1 + e_1 = x_1 \Rightarrow e_1 = 0$

$\therefore y_1 + e_2 = y_1 \Rightarrow e_2 = 0$

$\therefore e = (e_1, e_2) = (0, 0)$

Thus, $U+e = e+U = e$

5 Let $i = (i_1, i_2)$ and $U = (x_1, y_1)$

$\therefore U + i = e$

$\therefore (x_1, y_1) + (i_1 + i_2) = (0, 0)$

$\therefore x_1 + i_1 = 0$

$\therefore y_1 + i_2 = 0$

$\therefore i = (i_1, i_2) = (-x_1, -y_1)$

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$\therefore U + I = I + U = e$

6 For scalar k,

$\therefore KU = K(x_1, y_1)$

$= (0, ky_1) \in R$

7 $K(U+V) = K[(x_1, y_1) + (x_2, y_2)]$

$= K[(x_1 + x_2) \cancel{+}, (y_1 + y_2)]$

$= K(x_1 + x_2), K(y_1 + y_2)$ — 1

$\Rightarrow KU + KV = K(x_1, y_1) + K(x_2, y_2)$

$= (0, ky_1) + (0, ky_2)$

$= (0, K(y_1 + y_2))$ — 2

Here, eq^n $1 \neq 2$

$\therefore K(U+V) = KU + KV$

Hence, R^2 is not a vector space under the given operation.

C $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$
and
$k(x_1, x_2) = (k^2 x_1, k^2 x_2)$

$\Rightarrow$ Let,
$U = (x_1, y_1)$ and $V = (x_2, y_2) \in R$
and $W = (x_3, y_3) \in R$.

For scaler k_1 and $k_2 \in R$.

1 $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in R^2$

2 $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1)$

$= [(x_2, y_2) + (x_1, y_1)]$

$U + V = V + U$

3 $U + (V + W) = U + ((x_2, y_2) + (x_3, y_3)]$

$= U + (x_2 + x_3, y_2 + y_3)$

$= (x_1, y_1) + (x_2 + x_3, y_1 + y_3)$

$u+(v+w) = (x_1+x_2+x_3,\ y_1+y_2+y_3)$ — ①

→ $(u+v)+w = [(x_1,y_1)+(x_2,y_2)]+w$

$= (x_1+x_2,\ y_1+y_2) + (x_3,\ y_3)$

$(u+v)+w = (x_1+x_2+x_3,\ y_1+y_2+y_3)$ — ②

By eq^n 1 = 2

$\therefore u+(v+w) = (u+v)+w$

4 Let $e = (e_1, e_2)$

Such that, $u+e = u$

$\therefore (x_1, y_1)+(e_1, e_2) = (x_1, y_1)$

$\therefore (x_1+e_1,\ y_1+e_2) = (x_1, y_1)$

$\therefore x_1+e_1 = x_1 \Rightarrow e_1 = 0$

$\therefore y_1+e_2 = y_1 \Rightarrow e_2 = 0$

$\therefore e = (e_1, e_2) = (0, 0)$

$\therefore u+e = e+u = e$

5 Let $I = (i_1, i_2)$ and $u = (x_1, y_1)$

$\therefore u+i = e$

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$\therefore (x_1, y_1) + (i_1, i_2) = (e_1, e_2)$

$\therefore (x_1, y_1) + (i_1, i_2) = (0, 0)$

$\therefore x_1 = -i_1, \; x_2 = -i_2$

$\therefore I = (i_1, i_2) = (-x_1, -y_1) \in R^2$

6 For scalar k

$\therefore kU = k(x_1, y_1)$

$= (k^2x_1, k^2y_1) \in R^2$

7 $k(U+V) = k((x_1, y_1) + (x_2, y_2))$

$= k(x_1+x_2, y_1+y_2)$

$= (k^2(x_1+x_2), k^2(y_1+y_2)) \quad - (1)$

$\Rightarrow kU + kV = k(x_1, y_1) + k(x_2, y_2)$

$= (k^2x_1, k^2y_1) + (k^2x_2, k^2y_2) \quad - (2)$

$= \cancel{(k^2x_1 +}$

By eq^n $1 \neq 2$

$\therefore k(U+V) \neq kU + kV$

Hence, R^2 is not a vector space under the given operation.

d $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$
and $k(x_1, x_2) = (kx_1, kx_2)$

=> Let $U = (x_1, y_1)$, $V = (x_2, y_2)$, $W = (x_3, y_3) \in R^2$
For k_1 and $k_2 \in R^2$

1 $V \in V$

$U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1) \in R^2$

2 $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1)$

$= (x_2, y_2) + (x_1, y_1)$

$U + V = V + U$

3 $U + (V + W) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$= (x_1, y_1) + (x_2 + x_3 + 1, y_2 + y_3 + 1)$

$= (x_1 + x_2 + x_3 + 2, y_1 + y_2 + y_3 + 2)$ — (1)

$\Rightarrow (U+V)+W = [(x_1, y_1)+(x_2, y_2)] + (x_3, y_3)$

$= (x_1+x_2+x_3+2,\ y_1+y_2+y_3+2) - ①$

By eq^n 1 = 2

4 Let $e = (e_1, e_2)$

Such that $U+e = U$

$\therefore (x_1, y_1)+(e_1, e_2) = (x_1, y_1)$

$\therefore (x_1+e_1+1,\ y_1+e_2+1) = (x_1, y_1)$

$\therefore x_1+e_1+1 = x_1 \Rightarrow e_1 = -1$

$\therefore y_1+e_2+1 = y_1 \Rightarrow e_2 = -1$

$\therefore e = (-1,-1) \Rightarrow U+e = U$

5 $I = (i_1, i_2)$

$\therefore U+I = e$

$\therefore (x_1, y_1)+(i_1, i_2) = (-1,-1)$

$\therefore (x_1+i_1+1,\ y_1+i_2+1) = (-1,-1)$

$\therefore x_1+i_1+1 = -1 \Rightarrow x_1 = 2-i_1$

$\therefore y_1+i_2+1 = -1 \Rightarrow x_2 = 2-i_2$

$\therefore I = (i_1, i_2) = (-2-x_1, -2-y_1) \in R^2$

6 For scalar k

$\therefore KU = K(x_1, y_1) = (kx_1, ky_1) \in R^2$

7 $K(U+V) = k((x_1, y_1) + (x_2, y_2))$

$= k[x_1 + x_2 + 1, \; y_1 + y_2 + 1]$

$= (kx_1 + kx_2 + k, \; ky_1 + ky_2 + k) \; - (1)$

$\Rightarrow KU + KV = k(x_1, y_1) + k(x_2, y_2)$

$= (kx_1, ky_1) + (kx_2, ky_2)$

$= (kx_1 + kx_2 + 1, \; ky_1 + ky_2 + 1) \; - (2)$

By eq^n $1 \neq 2$

Hence, R^2 is not a vector space for this operation.

(ii) The set of $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$

and $k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$

Let $A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$, $B = \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix}$, $C = \begin{bmatrix} e & 1 \\ 1 & f \end{bmatrix} \in M$

For scaler k_1 and $k_2 \in M$

1 $A+B = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix}$

$= \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix} \in M$

2 $A+B = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$

$= \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} + \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

$A+B = B+A$

3 $A+(B+C) = A + \left(\begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} + \begin{bmatrix} e & 1 \\ 1 & f \end{bmatrix} \right)$

$= \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c+e & 1 \\ 1 & d+f \end{bmatrix}$

$= \begin{bmatrix} a+c+e & 1 \\ 1 & b+d+f \end{bmatrix}$ — 1

$\rightarrow$ $(A+B)+C = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix} + \begin{bmatrix} e & 1 \\ 1 & f \end{bmatrix}$

$= \begin{bmatrix} a+c+e & 1 \\ 1 & b+d+f \end{bmatrix}$ — 2

By 1 = 2 $\Rightarrow$ $A+(B+C) = (A+B)+C$

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4 Let $e = \begin{bmatrix} e_1 & 1 \\ 1 & e_2 \end{bmatrix}$

$\therefore e + A = A \Rightarrow \begin{bmatrix} e_1 & 1 \\ 1 & e_2 \end{bmatrix} + \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

$\therefore \begin{bmatrix} e_1 + a & 1 \\ 1 & e_2 + b \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

$\therefore e_1 + a = a \Rightarrow e_1 = 0$
$\therefore e_2 + b = b \Rightarrow e_2 = 0$

$\therefore e = \begin{bmatrix} 0 & 1 \\ \cancel{0}1 & 0 \end{bmatrix} \in M \Rightarrow A + e = A$

5 $I = \begin{bmatrix} i_1 & 1 \\ 1 & i_2 \end{bmatrix}$

$\therefore A + I = e \Rightarrow \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} i_1 & 1 \\ 1 & i_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} a + i_1 & 1 \\ 1 & b + i_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore a + i_1 = 0 \Rightarrow i_1 = -a$
$\therefore b + i_2 = 0 \Rightarrow i_2 = -b$

$I = \begin{bmatrix} -a & 1 \\ 1 & -b \end{bmatrix} \in M \Rightarrow A + I = e$

6 $KA = \begin{bmatrix} Ka & 1 \\ y & Kb \end{bmatrix}$ GM

7 $K(A+B) = K\begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$

$= \begin{bmatrix} K(a+c) & 1 \\ 1 & K(b+d) \end{bmatrix}$ — (1)

$\rightarrow KA + KB = \begin{bmatrix} Ka & 1 \\ 1 & Kb \end{bmatrix} + \begin{bmatrix} KC & 1 \\ 1 & Kd \end{bmatrix}$

$= \begin{bmatrix} K(a+c) & 1 \\ 1 & K(b+d) \end{bmatrix}$ — (2)

From eq^n 1 = 2

$\therefore K(A+B) = KA + KB$

8 $(K_1+K_2)A = (K_1+K_2)\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

$= \begin{bmatrix} a(K_1+K_2) & 1 \\ 1 & b(K_1+K_2) \end{bmatrix}$ — (1)

$\rightarrow K_1A + K_2A = K_1\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + K_2\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

$= \begin{bmatrix} K_1a & 1 \\ \cancel{y}1 & K_1b \end{bmatrix} + \begin{bmatrix} K_2a & 1 \\ 1 & K_2b \end{bmatrix}$

$$= \begin{bmatrix} a(k_1+k_2) & 1 \\ 1 & b(k_1+k_2) \end{bmatrix} \quad \text{— } (2)$$

From eq^n $1 = 2 \Rightarrow (k_1+k_2)A = k_1A + k_2A$

9 $k(mU) = k\begin{bmatrix} ma & 1 \\ 1 & mb \end{bmatrix} = \begin{bmatrix} kma & 1 \\ 1 & kmb \end{bmatrix}$

$\therefore k(mU) = km(U)$

10 $1 \cdot U = 1 \cdot \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

Hence, M is a vector space under the operation.

(iii) Check wether the set of all pair of real numbers of form $(1, x)$ with operation, $(1, x) + (1, y) = (1, x+y)$ and $k(1, x) = (1, kx)$

$\Rightarrow$ Let $U = (1, y_1)$, $V = (1, y_2)$, $W = (1, y_3) \in R$ and k_1, k_2 are scaler.

1 $U + V = (1, y_1) + (1, y_2)$

$= (1, y_1 + y_2) \in R$

2 $U + V = (1, y_1 + y_2)$

$= (1, y_2 + y_1)$

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$= (1, y_2) + (1, y_1)$

$U + V = V + U$

3 $(U+V)+W = (1, y_1 + y_2) + (1, y_3)$

$= (1, y_1 + y_2 + y_3) \in R$ — (1)

-> $U + (V+W) = (1, y_1) + ((1, y_2) + (1, y_3))$

$= (1, y_1) + (1, y_2 + y_3)$

$= (1, y_1 + y_2 + y_3)$ — (2)

From eq^n 1 = 2 => $(U+V)+W = U+(V+W)$

4. Let $e = (1, e_2)$

Such that $U + e = U$ => $(1, y_1) + (1, e_2) = (1, y$

$\therefore (1, y_1 + e_2) = (1, y_1)$

$\therefore y_1 + e_2 = y_1$ => $e_2 = 0$

$e = (1, 0)$ => $U + e = U$

5 Let $I = (1, i_2)$

Such that $(U) + I = e$

$\therefore (1, y_1) + (1, i_2) = (1, 0)$

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$\therefore y_1 + i_2 = 0 \Rightarrow i_2 = -y_1$

$\therefore I = (1, -y_1) \in R$

6 For scaler k

$kU = k(1, y_1) = (1, ky_1) \in R$

7 $k(U+V) = k[(1, y_1+y_2)]$

$= \cancel{k}(1, k(y_1+y_2)) \in R$ — (1)

$\rightarrow kU + kV = k(1, y_1) + k(1, y_2)$

$= (1, ky_1) + (1, ky_2)$

$= (1, k(y_1+y_2))$ — (2)

From eq^n 1 = 2 $\Rightarrow k(U+V) = kU + kV$

8 $(k_1+k_2)U = (k_1+k_2)(1, y_1)$

$= (1, (k_1+k_2)y_1)$ — (1)

$\rightarrow k_1U + k_2U = k_1(1, y_1) + k_2(1, y_1)$

$= (1, k_1y_1) + (1, k_2y_1)$

$= (1, (k_1+k_2)y_1)$ — (2)

from eq^n 1 = 2

$\therefore (k_1 + k_2)U = k_1U + k_2U$

9 $k(mU) = k(1, my_1)$

$= (1, kmy_1)$

$= km(1, y_1)$

$k(mU) = km(U)$

10 $1 \cdot U = 1 \cdot (1, y_1) = (1, 1y_1)$

Hence, R^2 is a vector space for all the operation.

(iv) Check wether the set of real number $(1, x)$ with operaction,

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

and $k(x_1, x_2) = (kx_1, kx_2)$ is vector space or not.

→ Let $U = (x_1, y_1)$, $V = (x_2, y_2)$ and $W = (x_3, y_3) \in R^2$

and k_1, k_2 are scaler.

1 $U + V = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in R^2$

2 $U+V = (x_1+x_2, \ y_1+y_2)$

$= (x_2+x_1, \ y_2+y_1)$

$U+V = (V+U) \in R^2$

3 $U+(V+W) = U+(x_2, y_2)+(x_3, y_3)$

$= (x_1, y_1)+(x_2+x_3, \ y_2+y_3)$

$= (x_1+x_2+x_3, \ y_1+y_2+y_3)$ — 1

→ $(U+V)+W = (x_2+x_1, \ y_2+y_1)+(x_3, y_3)$

$= (x_1+x_2+x_3, \ y_1+y_2+y_3)$ — 2

From eq^n 1 = 2 ⇒ $U+(V+W) = (U+V)+W$

4 Let $e = (e_1, e_2)$

∴ $U+e = U$ ⇒ $(x_1, y_1)+(e_1, e_2) = (x_1, y_1)$

∴ $(x_1+e_1, \ y_1+e_2) = (x_1, y_1)$

∴ $x_1+e_1 = x_1$ ⇒ $e_1 = 0$

∴ $y_1+e_2 = y_1$ ⇒ $e_2 = 0$

∴ $e = (e_1, e_2) = (0, 0)$

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5 Let $I = (i_1, i_2)$

Such that $U + I = e$

$\therefore (x_1, y_1) + (i_1, i_2) = e$

$\therefore (x_1 + i_1, y_1 + i_2) = e$

$\therefore (x_1 + i_1, y_1 + i_2) = (0, 0)$

$\therefore x_1 + i_1 = 0 \Rightarrow i_1 = -x_1$

$\therefore y_1 + i_2 = 0 \Rightarrow i_2 = -y_1$

$\therefore I = (-x_1, -y_1)$

$\therefore U + I = e = I + U$

6 For scaler k

$kU = k(x_1, y_1)$

$= (kx_1, ky_1) \in R^2$

7 $k(U+V) = k((x_1, y_1) + (x_2, y_2))$

$= k(x_1 + x_2, y_1 + y_2)$

$= (kx_1 + kx_2, ky_1 + ky_2)$ — (1)

$\rightarrow$ $KU + KV = K(x_1, y_1) + K(x_2, y_2)$

$= (Kx_1, Ky_1) + (Kx_2, Ky_2)$

$= (Kx_1 + Kx_2, Ky_1 + Ky_2)$ — 2

From eq^n 1 = 2

$\therefore K(U+V) = KU + KV$

8 $(K_1 + K_2)U = (K_1 + K_2)(x_1, y_1)$

$= (k_1 x_1 + k_2 x_1, k_2 y_1 + k_1 y_1)$ — 1

$\rightarrow$ $K_1 U + K_2 U = K_1(x_1, y_1) + K_2(x_1, y_1)$

$= (k_1 x_1, k_1 y_1) + (k_2 x_1, k_2 y_1)$

$= (k_1 x_1 + k_2 x_1, k_1 y_1 + k_2 y_1)$ — 2

From eq^n 1 = 2

$\therefore K_1 U + K_2 U = (K_1 + K_2)U$

9 $K(mU) = K(m(x_1, y_1))$

$= K(mx_1, my_1)$

$= (Kmx_1, Kmy_1)$

$= km(x_1, y_1)$

$k(mU) = km(U)$

10 $1 \cdot U = 1 \cdot (x_1, y_1)$

$= (x_1, y_1)$

=> Hence, R^2 is a vector space for all the operation.

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* Task : 2 Subspace.

(i) Check wether the following are subspace of R^2 or R^3.

a $W = \{(x, y, z) \in R^3 \mid x^2+y^2+z^2 \leq 1\}$

=) Let $U = (x_1, y_1, z_1)$ and
$V = (x_2, y_2, z_2) \in W$

Where $x_1^2+y_1^2+z_1^2 \leq 1$ and $x_2^2+y_2^2+z_2^2 \leq 1$

-) Properties : 1

$U+V = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1+x_2, y_1+y_2, z_1+z_2)$

but, Here, $x^2+y^2+z^2 = 1$

$\therefore (x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2$

$\therefore x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2$
$+ z_1^2 + 2z_1z_2 + z_2^2$

$\therefore 2x_1x_2 + 2y_1y_2 + 2z_1+z_2 + 2 \neq 1$

So, W is not Sub space of R^3.

b $W = \{(x, y, z) \in R^3 / ax + by + cz = 0, a, b, c \in R\}$

=> Let, $U = (x_1, y_1, z_1)$ and
$V = (x_2, y_2, z_2) \in R^3$

Here, $ax_1 + by_1 + cz_1 = 0$ and
$ax_2 + by_2 + cz_2 = 0$

-> Properties : 1

$U + V = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

where, $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2)$

$\therefore ax_1 + ax_2 + by_1 + by_2 + cz_1 + cz_2$

$= 0$

Hence, $U + V \in W$

-> Properties : 2

$KU = K(x_1, y_2, z_1)$

$= kax_1 + kb\cancel{y_2} + kcz_1$

$= 0$

$KU \in W.$

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So, W is sub space of R^3.

c $W = \{(x, y) \in R^2 / x^2 = y^2\}$

=) Let $U = (-1, -2)$ and $V = (-3, 4) \in W$

-) Properties : 1

$U + V = (-1, -2) + (-3, 4)$

$= (-4, 2) \in W$

$\because$ Here, $16 \neq 4$

So, W is not Sub space of R^2.

d $W = \{(x, 0, 0) \in R^3 \mid x \in R\}$

=) Let ~~U = (x,, 0~~ $U = (x_1, 0, 0)$ and
$V = (x_2, 0, 0) \in W$

-) Properties : 1

$U + V = (x_1, 0, 0) + (x_2, 0, 0)$
$= (x_1 + x_2, 0, 0)$

Here, $x_1 \in R$ and $x_2 \in R$
So, $x_1 + x_2 \in R$

$\therefore U + V \in W$

$\rightarrow$ Properties : 2

$\rightarrow$ $KU = K(x_1, 0, 0)$

$= (Kx_1, 0, 0) \in R$

$\therefore KU \in W$

So, W is a Subspace of R^3

(ii) Check wether $W = \{A \in M_{22}, \det(A) \neq 0\}$ is subspace of M_{22}

$\Rightarrow$ Let $U = \{A_1 \in M_{22}, \det(A_1) \neq 0\}$
$V = \{A_2 \in M_{22}, \det(A_2) \neq 0\}$

$\rightarrow$ Properties : 1

$U + V = A_1 + A_2$

Here, $\det(A_1) \neq 0$
$\det(A_2) \neq 0$

So, $U + V \in W$

$\rightarrow$ Properties : 2

$KU = KA_1$

$K\det(A_1) \neq 0$

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So, $KU \in W$

So, W is subspace of M_{22}

(iii) Check wether $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}, \text{ where } a+b+c+d=0 \right\}$ is subspace of M_{22}.

=) Let $U = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $V = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

where $a_1 + b_1 + c_1 + d_1 = 0$ and
$a_2 + b_2 + c_2 + d_2 = 0$

-) Properties : 1

$U + V = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

$= a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2$

$= 0$

$\therefore U + V \in W$

-) Properties : 2

$$KU = K\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$$= ka_1 + kb_1 + kc_1 + kd_1$$

$KU = 0$

$\therefore KU \in W$

W is Subspace of M_{22}.

(iv) Check wether $W = \{a_3x^3 + a_2x^2 + a_1x + a_0 \in P_3 ;\ a_0 = 0\}$ is subspace of P_3

=) Let $U = a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0$

$V = b_3x_1^3 + b_2x_1^2 + b_1x_1 + b_0$

where $a_0 = 0$ and $b_0 = 0$

-) Properties: 1

$$U + V = a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0 + b_3x_1^3 + b_2x_1^2 + b_1x_1 + b_0$$

$$= x_1^3(a_3 + b_3) + x_1^2(a_2 + b_2) + x_1(a_1 + b_1) + a_0 + b_0 \in W$$

$\therefore U + V \in W$

→ Properties : 2

$KU = Ka_3x_1^3 + a_2x_1^2K + Ka_1x_1 + Ka_0$

$\therefore KU = W$

So, W is a Subspace of P_3

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* Task : 3 Linear combination, Linearly Dependent and Independent

(i) Show that $V = (9, 2, 7)$ is a Linear combination of the vector $V_1 = (1, 2, -1)$ and $V_2 = (6, 4, 2)$.

Let $V = K_1 V_1 + K_2 V_2$

$\therefore (9, 2, 7) = K_1(1, 2, -1) + K_2(6, 4, 2)$

$\therefore (9, 2, 7) = (K_1, 2K_1, -K_1) + (6K_2, 4K_2, 2K_2)$

$\therefore 1K_1 + 6K_2 = 9$
$\therefore 2K_1 + 4K_2 = 2$
$\therefore -K_1 + 2K_2 = 7$

=> System of Augmented matrix.

$$\therefore \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right].$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right]$$

$R_2 \to R_2/-8$
$R_3 \to R_3/8$

$$= \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\to$ Here, $k_2 = 2$ and $k_1 + 6k_2 = 9$

$\therefore k_1 = -3$

$\to$ System is consistent.
Thus, V can be express as linear combination of V_1, V_2

(ii) Express a vector $V = (7, 4, -3)$ is a linear combination of vector $V_1 = (1, -2, -5)$ and $V_2 = (2, 5, 6)$.

$\Rightarrow$ Let, $V = k_1 V_1 + k_2 V_2$

$\therefore (7, 4, -3) = k_1(1, -2, -5) + k_2(2, 5, 6)$

$\therefore (7, 4, -3) = (k_1, -2k_1, -5k_1) + (2k_2, 5k_2, 6k_2)$

$\therefore \quad k_1 + 2k_2 = 7$

$\therefore \quad -2k_1 + 5k_2 = 4$

$\therefore \quad -5k_1 + 6k_2 = -3$

-> System of augmented matrix,

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 5R_1$

$$= \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right]$$

$R_2 \to R_2/9$

$R_3 \to R_3/16$

$$= \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

=> Here, $k_2 = 2 \Rightarrow k_1 + 2k_2 = 7$

$\therefore \quad k_1 = -1$

=> Hence, System is Consistent
So, We can express V in Linear combination of V_1 and V_2.

(iii) Express a vectors $P(x) = 9x^2 + 8x + 7$ is a linear combination of the vector $P_1 = 4x^2 + x + 2$ and $P_2 = 3x^2 - x + 1$ and $P_3 = 5x^2 + x + 2$

=> Let, $P = k_1 P_1 + k_2 P_2 + k_3 P_3$

$\therefore 9x^2 + 8x + 7 = k_1(4x^2 + x + 2) + k_2(3x^2 - x + 1) + k_3(5x^2 + x + 2)$

$\therefore 9x^2 + 8x + 7 = 4x^2 k_1 + x k_1 + 2k_1 + 3x^2 k_2 - x k_2 + k_2 + 5x^2 k_3 + x k_3 + 2k_3$

$\therefore 9x^2 + 8x + 7 = (4k_1 + 3k_2 + 5k_3)x^2 + (k_1 - k_2 + k_3)x + (2k_1 + k_2 + 2k_3)$

$\therefore \quad 9 = 4k_1 + 3k_2 + 5k_3$

$\therefore \quad 8 = k_1 - k_2 + k_3$

$\therefore \quad 7 = 2k_1 + k_2 + 2k_3$

=> System of Augmented Matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 3 & 5 & 9 \\ 1 & -1 & 1 & 8 \\ 2 & 1 & 2 & 7 \end{array}\right]$$

$$R_1 \longleftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 4 & 3 & 5 & 9 \\ 2 & 1 & 2 & 7 \end{array}\right]$$

$R_2 \rightarrow R_2 - 4R_1$
$R_3 \rightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 7 & 1 & -23 \\ 0 & 3 & 0 & 9 \end{array}\right]$$

$R_2 \rightarrow R_2/7$
$R_3 \rightarrow R_3/3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & 1/7 & -23/7 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & 1/7 & -23/7 \\ 0 & 0 & -1/7 & 2/7 \end{array}\right]$$

$R_3 \rightarrow R_3(-7)$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & 1/7 & -23/7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\Rightarrow$ System is consistent.
So, $P(x)$ is express in linear combination of P_1, P_2 and P_3

(iv) Show that $V = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of the vector $V_1 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$ and

$V_3 = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$.

$\Rightarrow$ Let, $V = k_1V_1 + k_2V_2 + k_3V_3$

$$\therefore \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = k_1 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} + k_3 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} k_1 + 0k_2 + 4k_3 & -1k_1 + 2k_2 + 0k_3 \\ 2k_1 + 1k_2 - 2k_3 & 3k_1 + 4k_2 - 2k_3 \end{bmatrix}$$

$\therefore k_1 + 4k_3 = 6$
$\therefore -k_1 + 2k_2 = -8$
$\therefore 2k_1 + k_2 - 2k_3 = -1$
$\therefore 3k_1 + 4k_2 - 2k_3 = -8$

$\Rightarrow$ System of Augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ -1 & 2 & 0 & -8 \\ 2 & 1 & -2 & -1 \\ 3 & 4 & -2 & -8 \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -10 & -13 \\ 0 & 4 & -14 & -26 \end{array}\right]$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - 4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -12 & -12 \\ 0 & 0 & -22 & -12 \end{array}\right]$$

$R_3 \to R_3/-12$
$R_4 \to R_4/-22$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_4 \to R_4 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

=> Here, System is consistant
So, V is express a linear combination of V_1, V_2 and V_3

V Which of the following vector are linearly depent or Indepent.

(i) $V_1 = (-2, 0, 1)$, $V_2 = (3, 2, 5)$, $V_3 = (6, -1, 1)$
$V_4 = (7, 0, -2)$

=> Let, $k_1 V_1 + k_2 V_2 + k_3 V_3 + k_4 V_4 = 0$

$\therefore 0 = k_1(-2, 0, 1) + k_2(3, 2, 5) + k_3(6, -1, 1) + k_4(7, 0, -2)$

$\therefore -2k_1 + 3k_2 + 6k_3 + 7k_4 = 0$

$\therefore 2k_2 - 1k_3 = 0$

$\therefore 1k_1 + 5k_2 + k_3 - 2k_4 = 0$

-> System of Augmented matrix.

$$[A|B] = \left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right]$$

$$R_1 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & 3 & 6 & 7 & 0 \end{array}\right]$$

$$R_3 \to R_3 + 2R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array}\right]$$

$$R_2/2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 13R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$R_3/3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 1 & -2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$\therefore k_1 + 5k_2 + k_3 - 2k_4 = 0$

$\therefore k_2 - \frac{k_3}{2} = 0$

$\therefore k_3 + k_4 = 0$

$\rightarrow$ Here, we get equations but not any elements as zero.

So, V_1, V_2, V_3 and V_4 are linearly dependent.

(ii) $V_1 = (0,0,2,2)$, $V_2 = (3,3,0,0)$, $V_3 = (1,1,0,-1)$

$\Rightarrow$ Let, $k_1V_1 + k_2V_2 + k_3V_3 = 0$

$\therefore k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = 0$

$\therefore 3k_2 + k_3 = 0$

$\therefore 3k_2 + k_3 = 0$

$\therefore 2k_1 = 0$

$\therefore 2k_1 - k_3 = 0$

number of eq^n > number of unknow.

So, S is a linearly independent.

(iii) $V_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$,

$V_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$

Let, $\underline{k_1 V_1 + k_2 V_2 + k_3 V_3 + k_4 V_4} = 0$

$\therefore k_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = 0$

$\therefore \begin{bmatrix} k_1 & 2k_1 - k_2 + 2k_3 \\ 1k_1 - k_2 + 3k_3 - k_4 & -2k_1 + k_3 + 2k_4 \end{bmatrix} = 0$

$\therefore k_1 = 0$

$\therefore 2k_1 - k_2 + 2k_3 = 0$

$\therefore k_1 - k_2 + 3k_3 - k_4 = 0$

$\therefore -2k_1 + k_3 + 2k_4 = 0$

$\Rightarrow$ $\underline{\text{Augmented System}}$ matrix

$[A \mid B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 \\ 1 & -1 & 3 & -1 & 0 \\ -2 & 0 & 1 & 2 & 0 \end{array}\right]$

$\underline{R_1 \leftrightarrow R_3}$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 2 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + 2R_1$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & -2 & 7 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 + 2R_2$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 4 & 0 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$R_4 \to R_4/3$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore k_4 = 0$

$\therefore k_3 - k_4 = 0 \Rightarrow k_3 = 0$

$\therefore k_2 - 4k_3 + 2k_4 \Rightarrow k_2 = 0$

$\therefore k_1 - k_2 + 3k_3 - k_4 \Rightarrow k_1 = 0$

→ Here, we get eq^n elements are zero. So, given vectors are linearly independent.

(iv) $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $V_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

⇒ Let, $k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4 = 0$

$$\therefore k_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + k_4 \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} = 0$$

$\therefore k_1 + k_3 + 2k_4 = 0$

$\therefore 3k_2 + k_3 + 6k_4 = 0$

$\therefore k_2 + k_3 + 4k_4 = 0$

$\therefore 2k_1 + 2k_2 + 2k_3 + 6k_4 = 0$

⇒ Augmented matrix

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 6 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 2 & 2 & 2 & 6 & 0 \end{array}\right]$$

$R_1 \to R_4$

$$= \left[\begin{array}{cccc|c} 2 & 2 & 2 & 6 & 0 \\ 0 & 3 & 1 & 6 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_1 \to R_1/2$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 3 & 1 & 6 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_4 \to R_4 - R_1$

$R_2/3$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1/3 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 1 & -1 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + R_2$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1/3 & 2 & 0 \\ 0 & 0 & 2/3 & 2 & 0 \\ 0 & 0 & 1/3 & 1 & 0 \end{array}\right]$$

$$R_3 \to R_3 \times 3/2$$
$$R_4 \to R_4 \times 3$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1/3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$R_4 \to R_4 - R_3$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1/3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, we can not all value zero.
So, vectors are linearly dependent.

(V) $P_1 = x^2 + x + 2$, $P_2 = 3x^2 + 2x + 2$
$P_3 = 2x^2 + x$

$\Rightarrow$ Let, $k_1 P_1 + k_2 P_2 + k_3 P_3 = 0$

$\therefore k_1(x^2 + x + 2) + k_2(3x^2 + 2x + 2) + k_3(2x^2 + x) = 0$

$\therefore x^2 k_1 + x k_1 + 2k_1 + 3x^2 k_2 + 2x k_2 + 2k_2 + 2x^2 k_3 + x k_3 = 0$

$\therefore k_1 + 3k_2 + 2k_3 = 0$
$\therefore k_1 + 2k_2 + k_3 = 0$
$\therefore 2k_1 + 2k_2 = 0$

$\Rightarrow$ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to (-1) R_2$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

=> Here, we can not get any zero value
So, Vector are linearly dependent.

(vi) $f_1(x) = x$, $f_2(x) = \sin(x)$

Let, $f_1'(x) = 1$
$f_2'(x) = \cos x$

-> $W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$

$W = x\cos x - \sin x$

-> For $(-1, 1)$ interval w is zero
So, W is linearly indepenent function.

(vii) $f_1 = x^2 e^x, \quad f_2 = x e^x, \quad f_3 = e^x$

$\Rightarrow$ Let, $f_1' = 2xe^x + x^2 e^x$

$f_1'' = 2e^x + 2xe^x + 2xe^x + x^2 e^x$

$\rightarrow \quad f_2' = e^x + xe^x$

$f_2'' = e^x + xe^x + e^x$

$\rightarrow \quad f_3' = e^x$

$f_3'' = e^x$

$$\rightarrow \quad W = \begin{vmatrix} x^2 e^x & xe^x & e^x \\ 2xe^x + x^2 e^x & e^x + xe^x & e^x \\ 2e^x + 2xe^x + 2xe^x + x^2 e^x & 2e^x + xe^x & e^x \end{vmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_2 \rightarrow R_2 - R_1$

$$= \begin{vmatrix} x^2 e^x & xe^x & e^x \\ 2xe^x & e^x & 0 \\ 2e^x + 2e^x x & e^x & 0 \end{vmatrix}$$

$$= e^x \left[e^x \cdot 2xe^x - e^x (2e^x + 2xe^x) \right]$$

$$W = -2e^{3x}$$

$\longrightarrow$ Linear ~~indep~~independent.

* Task. :4 Linear Span.

1 Determine whether the vectors
$V_1 = (2, -1, 3)$, $V_2 = (4, 1, 2)$,
$V_3 = (8, -1, 8)$ span vector span R^3.

Let, $A = (a, b, c) \in R^3$

$\therefore A = K_1V_1 + K_2V_2 + K_3V_3$

$\therefore (a, b, c) = (2K_1, -K_1, 3K_1) + (4K_2, K_2, 2K_2)$
$+ (8K_3, -K_3, 8K_3)$

=> System of equation,

$\therefore 2K_1 + 4K_2 + 8K_3 = a$
$\therefore -K_1 + K_2 - K_3 = b$
$\therefore 3K_1 + 2K_2 + 8K_3 = c$

-> Augmented matrix,

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 4 & 8 & a \\ -1 & 1 & -1 & b \\ 3 & 2 & 8 & c \end{array}\right]$$

$R_1 \leftrightarrow R_2 \rightarrow R_1(-1)$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -b \\ 2 & 4 & 8 & a \\ 3 & 2 & 8 & c \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -b \\ 0 & 6 & 6 & a+2b \\ 0 & 5 & 5 & c+3b \end{array}\right]$$

$$R_2/6, \quad R_3/5$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -b \\ 0 & 1 & 1 & a+2b/6 \\ 0 & 1 & 1 & c+3b/5 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -b \\ 0 & 1 & 1 & a+2b/6 \\ 0 & 0 & 0 & \frac{c-3b}{5} - \frac{a+2b}{6} \end{array}\right]$$

$\Rightarrow$ Here, System is not consistance. than V is does not span for R^3.

2 Check whether the vectors $V_1 = (2, 2, 0)$, $V_2 = (1, 0, 0)$, $V_3 = (3, 3, 3)$ span vector space R^3 or not.

Let, $A = (a, b, c) \in R^3$

$\therefore A = K_1V_1 + K_2V_2 + K_3V_3$

$\therefore (a, b, c) = (2k_1, 2k_1, 0) + (k_2, 0, 0) + (3k_3, 3k_3, 3k_3)$

=> System of equation,

$\therefore 2k_1 + k_2 + 3k_3 = a$
$\therefore 2k_1 + 3k_3 = b$
$\therefore 3k_3 = c$

$\rightarrow [A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 2 & 0 & 3 & b \\ 0 & 0 & 3 & c \end{array}\right]$

$R_1/2$

$\therefore = \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & a/2 \\ 2 & 0 & 3 & b \\ 0 & 0 & 3 & c \end{array}\right]$

$R_2 \rightarrow R_2 - 2R_1$

$= \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & a/2 \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 3 & c \end{array}\right]$

=> Here, System is consistance
So, Vector V is span of R^3.

3 Determine whether the vectors $P_1 = 2x^2 - x + 2$, $P_2 = 2x^2 - 2x + 2$ and $P_3 = 4x^2 - x + 5$ span vector space P_2

$\Rightarrow$ Let $A = ax^2 + bx + c \in P_2$

$\therefore A = k_1 P_1 + k_2 P_2 + k_3 P_3$

$$\therefore ax^2 + bx + c = 2k_1 x^2 - k_1 x + 2k_1 + 2k_2 x^2 - 2k_2 x + 2k_2 + 4k_3 x^2 - k_3 x + 5k_3$$

$\rightarrow$ System of equation,

$$\therefore 2k_1 + 2k_2 + 4k_3 = a$$
$$\therefore -k_1 - 2k_2 - k_3 = b$$
$$\therefore 2k_1 + 2k_2 + 5k_3 = c$$

$\rightarrow$ Augmented matrix,

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & a \\ -1 & -2 & -1 & b \\ 2 & 2 & 5 & c \end{array}\right]$$

$R_1 \rightarrow R_1/2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & a/2 \\ -1 & -2 & -1 & b \\ 2 & 2 & 5 & c \end{array}\right]$$

$R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 - 2R_1$

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$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & a/2 \\ 0 & -1 & 1 & b+a/2 \\ 0 & 0 & 1 & c-a \end{array}\right]$$

=> Here, System is consistance.
So, Vector V is span of P_2

4 Check the vector $V_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$V_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$ and $V_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$ span vector

space M_{22}.

=> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$

$\therefore A = K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} K_1 & 2K_1 \\ K_1 & -2K_1 \end{bmatrix} + \begin{bmatrix} 0 & -K_2 \\ -K_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2K_3 \\ 3K_3 & K_3 \end{bmatrix}$

$+ \begin{bmatrix} 0 & 0 \\ -K_4 & 2K_4 \end{bmatrix}$

-> System of equation,

$\therefore K_1 = a$

$\therefore 2K_1 - K_2 + 2K_3 = b$

$\therefore K_1 - K_2 + 3K_3 - K_4 = c$

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$\therefore -2k_1 + k_3 + 2k_4 = d$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 2 & -1 & 2 & 0 & b \\ 1 & -1 & 3 & -1 & c \\ -2 & 0 & 1 & 2 & d \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 + 2R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & -1 & 2 & 0 & b-2a \\ 0 & -1 & 3 & -1 & c-a \\ 0 & 0 & 1 & 2 & d+2a \end{array}\right]$$

$$R_2(-1)$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & -2 & 0 & -b+2a \\ 0 & -1 & 3 & -1 & c-a \\ 0 & 0 & 1 & 2 & d+2a \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & -2 & 0 & 2a-b \\ 0 & 0 & 1 & -1 & c-b+a \\ 0 & 0 & 1 & 2 & d+2a \end{array}\right]$$

Here, System is Consistance.
So, Vector V is span of M_{22}.

* Task : 5 : Basis, Reduction and Extension to Basis for Vector Space.

1 Determine whether the vectors, $V_1 = (1, -1, 1)$, $V_2 = (0, 1, 2)$, $V_3 = (3, 0, -1)$ form the basis vectors of R^3.

=) Let, $V = (a, b, c) \in R^3$

$\therefore \quad V = k_1 V_1 + k_2 V_2 + k_3 V_3$

$\therefore (a, b, c) = (k_1, -k_1, k_1) + (0, k_2, 2k_2) + (3k_3, 0, -k_3)$

-> System of equation,

$$\therefore \quad k_1 + 3k_3 = a$$
$$-k_1 + k_2 = b$$
$$k_1 + 2k_2 - k_3 = c$$

-> Augmented matrix,

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ -1 & 1 & 0 & b \\ 1 & 2 & -1 & c \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & 3 & b+a \\ 0 & 2 & -4 & c-a \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & 3 & b+a \\ 0 & 0 & -10 & c-a \end{array}\right]$$

$\Rightarrow$ Here, System is consistance.
So, Vector V is span of R^3

$\Rightarrow$ Consider $k_1 V_1 + k_2 V_2 + k_3 V_3 = 0$

$\to$ System of equation,

$$\therefore k_1 + 3k_3 = 0$$
$$\therefore -k_1 + k_2 = 0$$
$$\therefore k_1 + 2k_2 - k_3 = 0$$

$\to$ Augmented Matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

$$R_2 \to R_2 + R_1$$
$$R_3 \to R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right]$$

=> Here, System is consistance.
So, Vector V is Linear Independant.

-> Hence, S is basis of V for R^3.

2 Determine whether the set of vectors

$$B = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

From the basis vectors of M_{22}.

=> Let, $V_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$V_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$

$\therefore V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R.$

$V = K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4$

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$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} +$

$k_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$

=> System of Equation,

$\therefore k_1 = a$

$\therefore 2k_1 - k_2 + 2k_3 = b$

$\therefore k_1 - k_2 + 3k_3 - k_4 = c$

$\therefore -2k_1 + k_3 + 2k_4 = d$

-> Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 2 & -1 & 2 & 0 & b \\ 1 & -1 & 3 & -1 & c \\ -2 & 0 & 1 & 2 & d \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 + 2R_1$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & -1 & 2 & 0 & b-2a \\ 0 & -1 & 3 & -1 & c-a \\ 0 & 0 & 1 & 2 & d+2a \end{array}\right]$$

$$R_2 \to R_2(-1)$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & -2 & 0 & -b+2a \\ 0 & -1 & 3 & -1 & c-a \\ 0 & 0 & 1 & 2 & d+2a \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & -2 & 0 & 2a-b \\ 0 & 0 & 1 & -1 & c+a-b \\ 0 & 0 & -1 & 2 & d+2a \end{array}\right]$$

$$R_4 \to R_4 + R_3$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & -2 & 0 & 2a-b \\ 0 & 0 & 1 & -1 & c+a-b \\ 0 & 0 & 0 & 1 & c-b+d+3a \end{array}\right]$$

Here, System is consistance
So, Vector V is span of R^3.

Consider $k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4 = 0$

System of equation, $k_1 = 0$

$\therefore 2k_1 - k_2 + 2k_3 = 0$

$\therefore k_1 - k_2 + 3k_3 - k_4 = 0$

$\therefore -2k_1 + k_3 + 2k_4 = 0$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 \\ 1 & -1 & 3 & -1 & 0 \\ -2 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + 2R_1$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2(-1)$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$R_3 \to R_2 + R_3$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Here, System is consistance.
So, Vector V is Linear Independent.

$\Rightarrow$ Hence, S is basis of V for R^3.

3 Determine whether the set of vector
$B = \{-1+4x+2x^2, 4+6x+x^2, 5+2x-x^2\}$
form the basis vector of P_2.

$\Rightarrow$ Let $V = ax^2+bx+c \in P_2$

$V_1 = 2x^2+4x-1$

$V_2 = x^2+6x+4$

$V_3 = -x^2+2x+5$

$$V = k_1V_1 + k_2V_2 + k_3V_3$$

$\therefore ax^2+bx+c = 2k_1x^2+4k_1x-k_1+$
$k_2x^2+6k_2x+4k_2$
$-k_3x^2+2k_3x+5k_3$

$\rightarrow$ System of Equation, $2k_1+k_2-k_3 = a$
$4k_1+6k_2+2k_3 = b$
$-k_1+4k_2+5k_3 = c$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & a \\ 4 & 6 & 2 & b \\ -1 & 4 & 5 & c \end{array}\right]$$

$R_1 \to R_1/2$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & a/2 \\ 4 & 6 & 2 & b \\ -1 & 4 & 5 & c \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & a/2 \\ 0 & 4 & 0 & b-2a \\ 0 & 9/2 & -9/2 & c+a/2 \end{array}\right]$$

$R_2/4$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & a/2 \\ 0 & 1 & 0 & b-2a/4 \\ 0 & 9/2 & -9/2 & c+a/2 \end{array}\right]$$

$R_3 \to R_3 - \frac{9}{2} R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & a/2 \\ 0 & 1 & 0 & b-2a/4 \\ 0 & 0 & -9/2 & c+\frac{a}{2} - \frac{9}{2}\left(\frac{b-2a}{4}\right) \end{array}\right]$$

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Here, System is not consistance. So, Vector V is not span.

So, Vector is not Basis for V.

4 Reduce that $B = \{(1,0,0), (0,4,-3), (0,1,-1), (0,2,0)\}$ to obtain a basis for R^3.

Let $V_1 = (1,0,0)$
$V_3 = (0,4,-3)$
$V_2 = (0,1,-1)$
$V_4 = (0,2,0)$

$\Rightarrow$ $K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4 = 0$

$K_1(1,0,0) + K_3(0,4,-3) + K_2(0,1,-1)$

$+ K_4(0,2,0) = 0$

$\rightarrow$ System of equation,

$\therefore K_1 = 0$
$\therefore K_2 + 4K_3 + 2K_4 = 0$
$\therefore -K_2 - 3K_3 = 0$

$\rightarrow$ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -1 & -3 & 0 & 0 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

In this matrix, leading one in 1^{st}, 2^{nd} and 3^{rd} column.

So, Basis $= \{(1,0,0), (0,1,-1), (0,4,-3)\}$

Reduce the set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$ to obtain a basis for M_{22}.

Let, $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$V_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $V_4 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\therefore K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4 = 0$

$\therefore K_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + K_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + K_4 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0$

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→ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right]$$

$R_4 \to R_4 - R_1$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In this matrix, leading one in 1^{st}, 2^{nd} column.

So, Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

6 Find a standard basis vectors that can be added to $B = \{(1,-2,-2), (-1,2,3)\}$ to produce a Basis for R^3.

Let, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$,
$e_3 = (0,0,1)$

$S = \{V_1, V_2, e_1, e_2, e_3\} \in R^3$

Consider,

$$k_1 V_1 + k_2 V_2 + k_3 e_1 + k_4 e_2 + e_3 k_5 = 0$$

$$\therefore k_1(1,-2,-2) + k_2(-1,2,3) + k_3(1,0,0) + k_4(0,1,0) + k_5(0,0,1) = 0$$

→ Augmented Matrix,

$$[A|B] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$

$$= \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

that

duce

$$= \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3/2$

$$= \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & -0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right]$$

=0

In this matrix, leading 1 in 1^{st}, 2^{nd} and 3^{rd} column.

Hence, Basis are $\{v_1, v_2, e_1\}$

$\therefore$ Basis $= \{(1,-2,-2), (-1,2,3), (1,0,0)\}$

7 Find a standard basis vectors that can be added to $B = \{(0,1,-1), (0,0,2)\}$ to produce a basis for R^3.

Let, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$S = \{v_1, v_2, e_1, e_2, e_3\} \subset R^3$

Consider,

$k_1 v_1 + k_2 v_2 + k_3 e_1 + k_4 e_2 + k_5 e_3 = 0$

$\therefore k_1(0,1,-1) + k_2(0,0,2) + k_3(1,0,0)$
$+ k_4(0,1,0) + k_5(0,0,1) = 0$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$R_2/2$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

In this matrix, leading 1 in 1^{st}, 2^{nd} and 3^{rd} column.

Basis are $\{v_1, v_2, e_1\}$

Basis $= \{(0,1,-1), (0,0,2), (1,0,0)\}$

* Task : 6 Basis for Row Space, Column Space and Null Space.

(i) Find the basis for row and column space.

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + R_1$
$R_4 \to R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$$

$R_2 \to R_2 / -14$
$R_3 \to R_3 / 4$
$R_4 \to R_4 / -5$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Basis for Row Space $= \{(1, 4, 5, 6, 9), (0, 1, 1, 1, 2)\}$

$\Rightarrow$ Basis for Column Space $= \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix} \right\}$

$$B \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ -1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$

$$= \cancel{\begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ & & & & \\ & & & & \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 & 3 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$R_3 / -5$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -8/5 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 3R_2$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -8/5 \\ 0 & 0 & 0 & 0 & -24/5 \end{bmatrix}$$

=> Basis for Row Space = 1^{st} row, 3^{rd}, 4^{th} and 5^{th} row

=> Basis for Column Space = 1^{st}, 3^{rd}, 4^{th}, 5^{th} column.

C $\begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 + R_1$

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$$= \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Basis for Row Space $= \{(1,4,5,4), (0,1,-2,-6), (0,0,1,5)\}$

$\Rightarrow$ Basis for Column Space $= \{(1,2,2,-1), (4,9,9,-4), (5,8,9,-5)\}$

(ii) Determine a basis for the following solution spaces of the homogeneous system.

A
$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

$\to$ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$R_1/3$$

$$= \left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 5R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & -8/3 & -2/3 & -8/3 & 0 \end{array}\right]$$

$$R_2\left(\frac{-3}{8}\right)$$

$$= \left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ Here $x_3 = r$ and $x_4 = t$

$$x_1 + \frac{x_3}{3} + \frac{x_2}{3} + \frac{x_4}{3} = 0$$

$$\therefore x_2 + \frac{x_3}{4} + x_4 = 0 \to x_2 = \frac{-r}{4} - t$$

$$\therefore x_1 = \frac{-r}{4}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -r/4 \\ -r/4 - t \\ r \\ t \end{bmatrix}$$

$$X = r\begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= rV_1 + tV_2$$

$\Rightarrow$ $V_1 = \begin{bmatrix} -1/4 \\ 1/4 \\ 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

B $2x_1 - 4x_2 + x_3 + 2x_4 - 2x_5 - 3x_6 = 0$
$-x_1 + 2x_2 + 0x_3 + 0x_4 + x_5 - x_6 = 0$
$10x_1 - 4x_2 - 2x_3 + 4x_4 - 2x_5 + 4x_6 = 0$

$\rightarrow$ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccccc|c} 2 & -4 & 1 & 2 & -2 & -3 & 0 \\ -1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 10 & -4 & -2 & 4 & -2 & 4 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{cccccc|c} -1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 2 & -4 & 1 & 2 & -2 & -3 & 0 \\ 10 & -4 & -2 & 4 & -2 & 4 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1(-1)$$

$$= \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & -1 & 1 & 0 \\ 2 & -4 & 1 & 2 & -2 & -3 & 0 \\ 10 & -4 & -2 & 4 & -2 & 4 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 10R_1$$

$$= \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5 & 0 \\ 0 & 16 & -2 & 4 & 8 & -6 & 0 \end{array}\right]$$

$$R_3 \to R_3 + 8R_1$$

$$= \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5 & 0 \\ 0 & 0 & -2 & 4 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2$$

$$= \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 4 & 0 & -8 & 0 \end{array}\right]$$

$$R_3 \to R_3 / 4$$

$$= \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$= \left[\begin{array}{cccccc|c} +1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 16 & -2 & 4 & 8 & -6 & 0 \\ 0 & 0 & 1 & 2 & 0 & -5 & 0 \end{array}\right]$$

$R_2 \to R_2/16$

$$= \left[\begin{array}{cccccc|c} +1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1/8 & 1/4 & 1/2 & -3/8 & 0 \\ 0 & 0 & 1 & 2 & 0 & -5 & 0 \end{array}\right]$$

$\Rightarrow$ Here, $x_4 = r$, $x_5 = s$ and $x_6 = t$

$\to \quad x_3 + 2x_4 - 5x_6 = 0$

$\therefore \quad x_3 = 5t - 2r$

$\to \quad x_2 - \frac{x_3}{8} + \frac{x_4}{4} + \frac{x_5}{2} - \frac{3}{8}x_6 = 0$

$\therefore \quad x_2 = t - \frac{s}{2} - \frac{r}{2}$

$\to \quad x_1 - 2x_2 - x_5 + x_6 = 0$

$\therefore \quad x_1 = t - r$

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$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} t - r \\ t - \frac{5}{2} - r/2 \\ 5t - 2r \\ r \\ s \\ t \end{bmatrix}$$

$$X = r\begin{bmatrix} -1 \\ -1/2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= rV_1 + sV_2 + tV_3$$

$$\rightarrow V_1 = \begin{bmatrix} -1 \\ -1/2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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(iii) Find a basis of $B = \{(x_1, x_2, x_3, x_4) R^4 \mid x_1 + x_2 = 0, x_2 + x_3 = 0, x_3 + x_4 = 0\}$

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* Task : 7 Rand and Nullity :

(i) Find the rank and nullity of the matrix.

$$A \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

Number of column = 3

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 7R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & -1 & +19 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$R_1 \to R_2 + R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & \cancel{-38}\ 0 & 0 \end{array}\right]$$

→ Rank of matrix : 2

→ Nullity = 3 − 2 = 1

B $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

→ Number of column = 4

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right]$$

$R_2 \to R_2/-7$

$$= \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$$= \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ Rank(A) = 2

→ Nullity (A) = 4 − 2 = 2

c $A = \begin{bmatrix} 1 & 3 & 4 & -2 & 2 \\ 3 & 7 & 6 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 \\ 1 & 2 & -2 & 6 & 3 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$= \begin{bmatrix} 1 & 3 & 4 & -2 & -1 \\ 0 & -2 & -6 & 8 & 4 \\ 0 & -2 & -6 & 8 & 4 \\ 0 & -2 & -6 & 8 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$$= \begin{bmatrix} 1 & 3 & 4 & -2 & -1 \\ 0 & -2 & -6 & 8 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 / -2$

$$= \begin{bmatrix} 1 & 3 & 4 & -2 & -1 \\ 0 & 1 & 3 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Rank(A) = 2

→ Nullity (A) = Number of column − Rank (A)

= 5 − 2

= 3

(ii) Verify the dimension theorem for the follwing matrix.

$$A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 7R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 / -3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-> Here, leading 1 in 1^{st} and 2^{nd} ~~colu~~ row,

Row space = {(1, 2, 3), (4, 5, 6)}

-> Here, $x_3 = t$

$\therefore x_1 + 2x_2 + 3x_3 = 0$

$\therefore x_2 + 2x_3 = 0$

$\therefore \boxed{x_2 = -2t}$

$\therefore x_1 - 4t + 3t = 0$

$\therefore x_1 = t$

→ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$

$X = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = tV_1$

→ Here, we get V_1 is basis of vector of solution space.

$\therefore$ Nullity (A) = 1

→ Nullity (A) = Number of column − Rank (A)
= 3 − 2
= 1

Thus, Dimension theorem is verify.

B $A = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 4 \\ -1 & 0 & 2 & 1 \end{bmatrix}$

→ Augmented matrix,

$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ -1 & 0 & 2 & 1 & 0 \end{array}\right]$

$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 0 & 5 & 4 & 0 \end{array}\right]$$

$R_2 / 2$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 5 & 4 & 0 \end{array}\right]$$

$R_3 / 5$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 4/5 & 0 \end{array}\right]$$

→ $Rank(A) = 3$

→ Here, $x_4 = t$

→ $x_3 + x_4 \frac{4}{5} = 0$

$\therefore x_3 = \frac{-4t}{5}$

→ $x_2 + x_3 + 2x_4 = 0$

$\therefore x_2 = -2t + \frac{4}{5}t = \frac{-3}{5}t$

→ $x_1 + x_2 + 3x_3 + 3x_4 = 0$

$\therefore x_1 = +2t - \frac{4}{5}t + \frac{12}{5}t + t - 3t$

$= 0$

→ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/5\, t \\ -t/2 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -3/5 \\ -1/2 \\ 1 \end{bmatrix} t = V_1 t$

→ Basis of vector is V_1.

$\therefore$ nullity $(A) = 1$ — (*)

→ nullity (A) = number of column − Rank(A)

$= 4 - 3$

$= 1$ — (**)

→ By eq^n (*) and (**)

dimension theorem is Verifiy.

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6 $A = \begin{bmatrix} 1 & 2 & 4 & -2 & -1 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 3 & 2 & 4 & -2 \\ 1 & 3 & -2 & -4 & 3 \end{bmatrix}$

→ Augmented matrix,

$$[A|B] = \left[\begin{array}{ccccc|c} 1 & 2 & 4 & -2 & -1 & 0 \\ 3 & 5 & 2 & 4 & 1 & 0 \\ 2 & 3 & 2 & 4 & -2 & 0 \\ 1 & 3 & -2 & -4 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 2 & 4 & -2 & -1 & 0 \\ 0 & -1 & -10 & 10 & 4 & 0 \\ 0 & -1 & -6 & 8 & 0 & 0 \\ 0 & 1 & -6 & -2 & 4 & 0 \end{array}\right]$$

$R_4 \to R_4 + R_2$
$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 2 & 4 & -2 & -1 & 0 \\ 0 & -1 & -10 & 10 & 4 & 0 \\ 0 & 0 & 4 & -2 & -4 & 0 \\ 0 & 0 & 4 & 8 & 8 & 0 \end{array}\right]$$

$$R_2 \to \frac{R_2}{-1}$$

$$R_4 \to R_4 - R_1$$

$$= \left[\begin{array}{ccccc|c} 1 & 2 & 4 & -2 & -1 & 0 \\ 0 & 1 & 10 & -10 & -4 & 0 \\ 0 & 0 & 4 & -2 & -4 & 0 \\ 0 & 0 & 0 & 10 & 12 & 0 \end{array}\right]$$

$$R_3 \to R_3 / 4$$
$$R_4 \to R_4 / 10$$

$$= \left[\begin{array}{ccccc|c} 1 & 2 & 4 & -2 & -1 & 0 \\ 0 & 1 & 10 & -10 & -4 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 1 & 6/5 & 0 \end{array}\right]$$

$\to$ rank $(A) = 4$

Here, we take, $x_5 = t$

$\to$ $x_4 + \frac{6}{5} x_5 = 0$

$\therefore x_4 = \frac{-6}{5} t$

$\to$ $x_3 - \frac{x_4}{2} - x_5 = 0$

$$\therefore x_3 = t - \frac{3}{5}t = \frac{2t}{5}$$

$\rightarrow$ $x_2 + 10x_3 - 10x_4 - 4x_5 = 0$

$$\therefore x_2 = -10 \times \frac{2t}{5} - 10\left(\frac{6t}{5}\right) + 4t$$

$$= -12t$$

$\rightarrow$ $x_1 = t - \frac{12}{5}t - \frac{8}{5}t + 24t$

$$= 21t$$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 21t \\ -12t \\ 2/5\,t \\ -6/5\,t \\ t \end{bmatrix} = t\begin{bmatrix} 21 \\ -12 \\ 2/5 \\ -6/5 \\ 1 \end{bmatrix} = tV_1$$

$\rightarrow$ Basis of vector is V_1

$\therefore$ Nullity $= 1$ — (1)

$\rightarrow$ Nullity = Number of column − Rank(A)

$$= 5 - 4$$

$$= 1 \quad — (2)$$

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Here, eq^n 1 = 2

$\therefore$ Dimension theorem is Verifly.

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* Task : 8 : Change of Basis

(i) Find the coordinate matrix.

A $V = (1, 2, 3)$ and Basis
$B = \{(2, -1, 2), (1, 0, 1), (1, 1, -1)\}$

Basis $B = \{V_1, V_2, V_3\}$

where, $V_1 = (2, -1, 2)$
$V_2 = (1, 0, 1)$
$V_3 = (1, 1, -1)$

Let, $V = K_1 V_1 + K_2 V_2 + K_3 V_3$

$\therefore (1, 2, 3) = (2K_1, -K_1, 2K_1) + (K_2, 0, K_2) + (K_3, K_3, -K_3)$

-> System of equations,

$\therefore 2K_1 + K_2 + K_3 = 1$ — 1
$\therefore -K_1 + K_3 = 2$ — 2
$\therefore 2K_1 + K_2 - K_3 = 3$ — 3

-> divide eq^n 1 and 3

$\therefore \cancel{2K_1} + \cancel{K_2} + K_3 = 1$
$\therefore \cancel{2K_1} + \cancel{K_2} - K_3 = 3$
$\quad - \quad - \quad + \quad -$

$2K_3 = -2$

$\therefore \boxed{K_3 = -1}$

-> put K_3 value in eq^n 2

$\therefore -K_1 + K_3 = 2$

$\therefore -K_1 - 1 = 2$

$\therefore \boxed{K_1 = -3}$

-> put K_1 and K_3 value in eq^n 1

$\therefore 2K_1 + K_2 + K_3 = 1$

$\therefore 2(-3) + K_2 - 1 = 1$

$\therefore K_2 = 1 + 1 + 6$

$\therefore K_2 = 8$

-> Coordinate matrix $[V]_S = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ -1 \end{bmatrix}$

B $V = 1 + x + x^2 + x^3$ and Basis

$B = \{1, x - x^2, 1 + x^2, x + x^3\}$

Basis $B = \{V_1, V_2, V_3, V_4\}$

where, $V_1 = (1, 0, 0, 0)$, $V_4 = (0, 1, 0, 1)$

$V_2 = (0\ 1, -1, 0)$

$V_3 = (1, 0, 1, 0)$

Let, $V = k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4$

$\therefore (1,1,1,1) = (k_1,0,0,0) + (0,k_2,-k_2,0) + (k_3,0,k_3,0) + (0,k_4,0,k_4)$

-> System of equation,

$\therefore k_1 + k_3 = 1$

$\therefore k_2 + k_4 = 1$

$\therefore k_3 - k_2 = 1$

$\therefore k_4 = 1$

and $k_2 = 0$

$k_3 = 1$

$k_1 = 0$

-> Coordinate matrix $[V]_S = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

C $V = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and Basis

$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Basis $B = \{V_1, V_2, V_3, V_4\}$

$k_4 V_4$

$(-k_2, 0)$

$(k_4, 0, k_4)$

where $V_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$V_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

→ Let, $V = k_1 V_1 + k_2 V_2 + k_3 V_3 + k_4 V_4$

$$\therefore \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -k_1 & k_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_2 & 0 \end{bmatrix} + \begin{bmatrix} k_3 & k_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k_4 \end{bmatrix}$$

→ System of equation,

$\therefore -k_1 + k_3 = 2$

$\therefore k_1 + k_3 = 0$

$\therefore k_2 = -1$

$k_4 = 3$

$\therefore k_3 = 1$ and $k_1 = -1$

→ Coordinate matrix, $[V]_S = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

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(ii) Consider the bases $B = (U_1, U_2)$ and $B' = (V_1, V_2)$ of R^2,

where $U_1 = (1, -1)$, $U_2 = (0, 6)$
$V_1 = (2, 1)$, $V_2 = (-1, 4)$
then Find the transition matrix from B to B'.

=> Let, $U_1 = k_1 V_1 + k_2 V_2$

$\therefore (1, -1) = k_1 (2, 1) + k_2 (-1, 4)$

$\therefore 2k_1 - k_2 = 1$ — (1) $\times 4$
$\therefore k_1 + 4k_2 = -1$ — (2)

$$\therefore \begin{array}{r} 8k_1 - \cancel{4k_2} = 4 \\ k_1 + \cancel{4k_2} = -1 \\ \hline 9k_1 = 3 \end{array}$$

$\therefore k_1 = \frac{1}{3}$

put k_1 value in eq^n 1

$\therefore 2 \times \frac{1}{3} - k_2 = 1$

$\therefore k_2 = \frac{-1}{3}$

-> $[a_1]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$

$\Rightarrow$ Let $U_2 = C_1V_1 + C_2V_2$

$\therefore (0, 6) = C_1(2, 1) + C_2(-1, 4)$

$\therefore 2C_1 - C_2 = 0 \quad - (1) \quad \times 4$

$\therefore C_1 + 4C_2 = 6 \quad - (2)$

$\therefore 8C_1 - \cancel{4C_2} = 0$

$C_1 + \cancel{4C_2} = 6$

$9C_1 = 6$

$\therefore C_1 = \frac{2}{3}$

put C_1 value in eq^n 1

$\therefore 2 \times \frac{2}{3} - C_2 = 0$

$\therefore C_2 = \frac{4}{3}$

$$[u_2]_B = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix}$$

$\rightarrow$ the transition matrix from B to B' $= \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 4/3 \end{bmatrix}$

(iii) Consider the bases $B=$
$B=\{(1,0,0),(0,1,0),(0,0,1)\}$ and
$B'=\{(1,-1,1),(0,1,2),(3,0,-1)\}$ of R^3

then find the transition matrix from B' to B and compute $[U]_B$, given that $[U]_{B'}=\begin{bmatrix}-2\\3\\4\end{bmatrix}$.

Let, the transition matrix from B' to B

$$P=\left[\,[V_1]B' \ [V_2]B' \ [V_3]B'\,\right]$$

→ $[V_1]B'=\begin{bmatrix}K_1\\K_2\\K_3\end{bmatrix}$

$\therefore V_1 = K_1U_1 + K_2U_2 + K_3U_3$

$\therefore (1,-1,1) = (K_1, K_2, K_3)$

$\therefore [V_1]B' = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

→ Similarlily,

$[V_2]B' = \begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}$

$\therefore V_2 = C_1U_1 + C_2U_2 + C_3U_3$

$\therefore (0,1,2) = C_1(1,0,0) + C_2(0,1,0)$
$+ C_3(0,0,1)$

$\therefore (0,1,2) = (C_1, C_2, C_3)$

$\therefore [V_2]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

→ Similarlily;

$[V_3]_{B'} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$

$\therefore V_3 = m_1U_1 + m_2U_2 + m_3U_3$

$\therefore (3,0,-1) = (m_1, m_2, m_3)$

$\therefore [V_3]_{B'} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

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$\Rightarrow \quad [U]_{B'} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$

$U = r_1V_1 + r_2V_2 + r_3V_3$

$\therefore U = (-2)(1,-1,1) + 3(0,1,2) + 4(3,0,-1)$

$\therefore U = (10, 5, 0)$

$[U]_B = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$