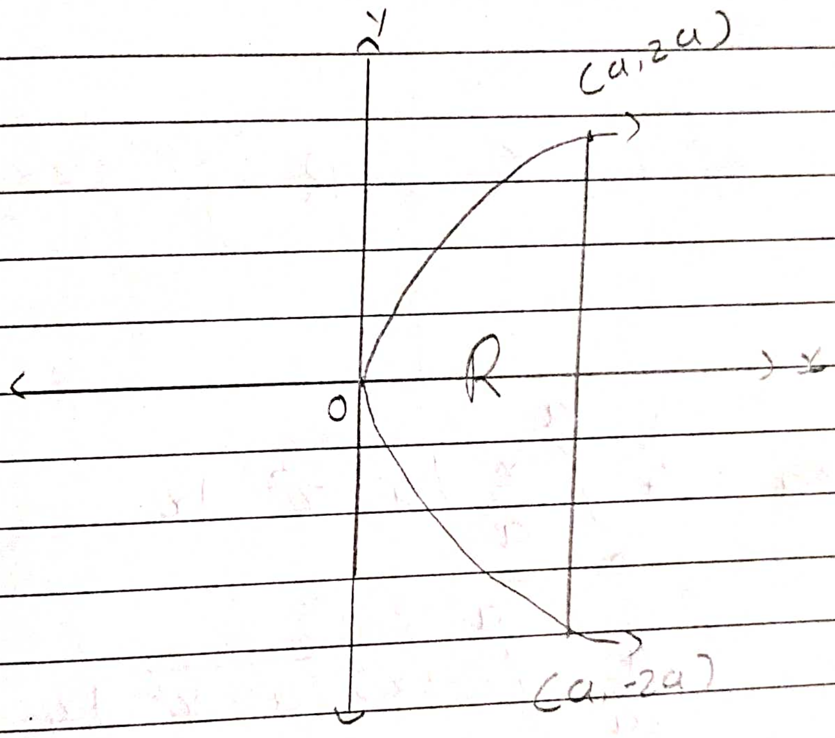
Unit - 5 Integral Calculus

* Task 1 : Area

1 Find the area between the parabola $y^2 = 4ax$ and it's latus rectum.



$$\text{Area} = 2\int_0^a 2\sqrt{ax} \cdot dx$$

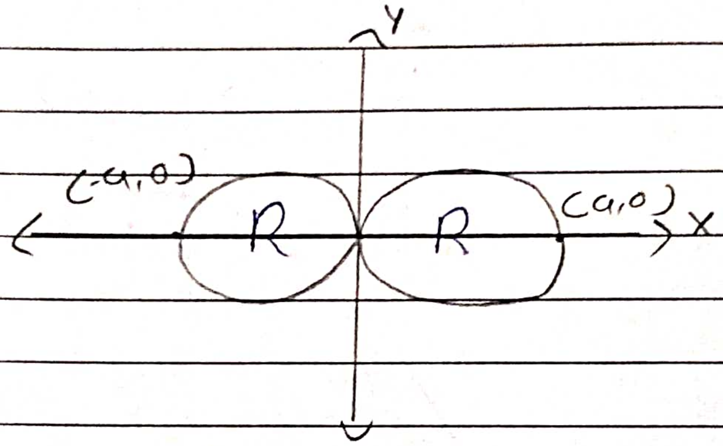
$$= 4\int_0^a \sqrt{a} \cdot x^{1/2}\, dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]$$

$$\text{Area} = \frac{8}{3}\sqrt{a}\cdot a^{3/2}$$

$$A = \frac{8}{3}a^{3}$$

2 Find the area of the loops of the curve $a^2y^2 = x^2(a^2 - x^2)$



$$\text{Area} = 4\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx$$

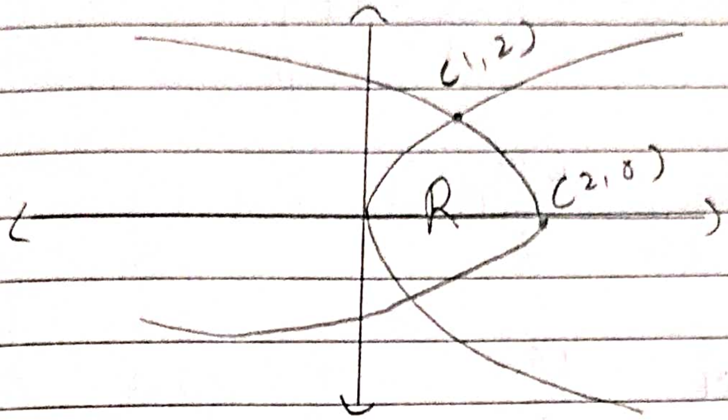
$$= \frac{-4}{2a}\int_0^a (-2x)\sqrt{a^2 - x^2}\,dx$$

$$= \frac{-2}{a}\left[\frac{2(a^2 - x^2)^{3/2}}{3}\right]_0^a$$

$$= \frac{-2}{a}\left[\frac{2}{3}(-a^2)^{3/2}\right]$$

$$\text{Area} = \frac{4a^3}{3}$$

3 Find the area enclosed by $y^2 = 4x$ and $y^2 = -4(x-2)$



$$x = \frac{y^2}{4} \quad \text{and} \quad x = \frac{8-y^2}{4}$$

$$\text{Area} = 2\int_{x=0}^{2}\int_{y=\frac{8-y^2}{4}}^{y^2/4} 1\, dx\, d\cancel{x}$$

$$= 2\int_0^2 [x]_{8-y^2/4}^{y^2/4}\, dy$$

$$= 2\int_0^2 \frac{y^2}{4} - \frac{8+y^2}{4}\, dy$$

$$= \frac{1}{2}\int_0^2 2y^2 - 8\, dy$$

$$= \frac{1}{2}\left[\frac{2y^3}{3} - 8y\right]_0^2$$

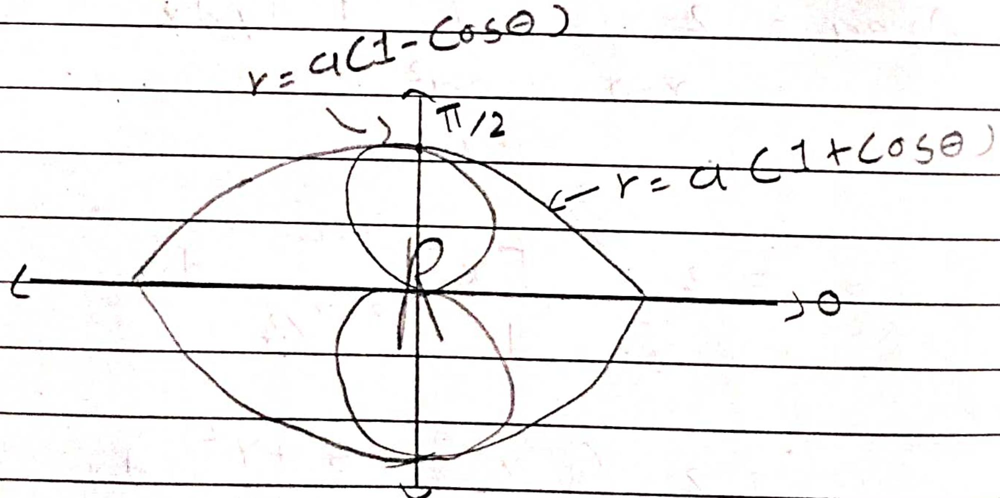
$$\text{Area} = \frac{1}{2}\left[\frac{2(8)}{3} - 8(2)\right]$$

$$= \frac{1}{2}\left[\frac{16}{3} - 16\right]$$

$$= \frac{+32}{6}$$

$$\text{Area} = \frac{16}{3}$$

4 Find the area between the the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$



$$\text{Area} = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a(1-\cos\theta)} r \, dr \, d\theta$$

$$= 4\int_{0}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= 4\int_0^{\pi/2} \frac{1}{2} a^2 (1-\cos\theta)^2 \, d\theta$$

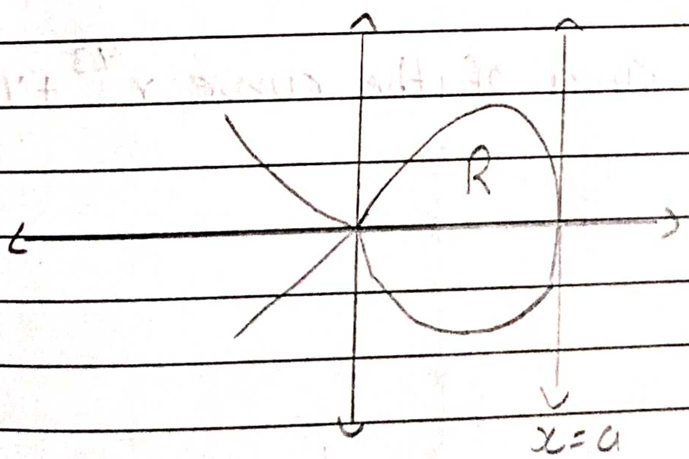
$$= 2a^2 \int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$= 2a^2 \left[\theta - 2\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{2a^2}{2}\left[\frac{3\pi}{4} - 2\right]$$

$$\text{Area} = a^2\left[\frac{3\pi}{2} - 4\right]$$

5 Find the area of the loop of $ay^2 = x^2(a-x)$ in first quadrant only.



$$\therefore \quad y = x\sqrt{\frac{a-x}{a}}$$

$$A = \int_0^a x \sqrt{\frac{a-x}{a}} \, dx$$

$$A = \frac{1}{\sqrt{a}} \int_0^a (a-(a-x))(a-x)^{1/2} \, dx$$

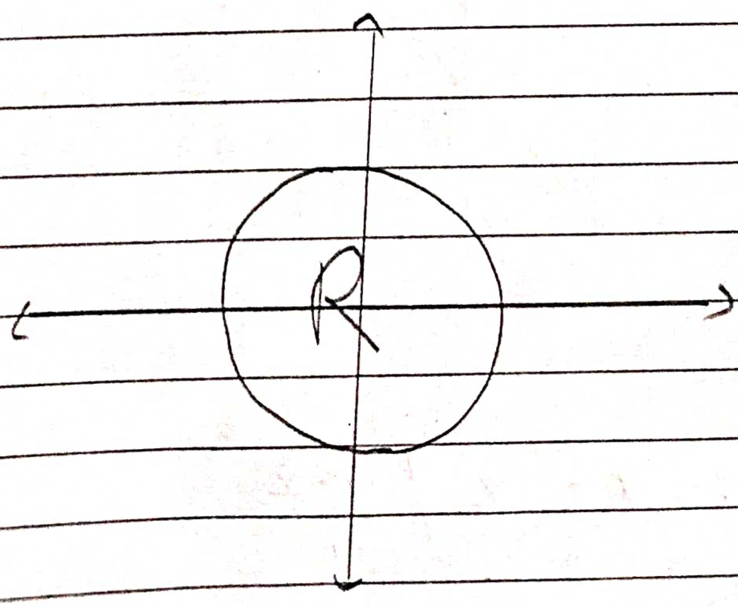
$$A = \frac{1}{\sqrt{a}} \int_0^a a\sqrt{a-x} - (a-x)^{5/2} \, dx$$

$$A = \frac{1}{\sqrt{a}} \left[\frac{-2a}{3}(a-x)^{3/2} + \frac{2}{5}(a-x)^{5/2} \right]_0^a$$

$$= \frac{1}{\sqrt{a}} \left[\frac{2}{3} a^{5/2} - \frac{2}{5} a^{5/2} \right]$$

$$= \frac{a^2 4}{15}$$

6 Find the area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$



Page No.
Date: / /

Here, $x = a\cos^3\theta$, $y = a\sin^3\theta$

$\therefore dx = -3a\cos^2\theta\sin\theta$

$$A = 4\int_0^{\pi/2} y\,dx$$

$$= 4\int_0^{\pi/2} a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$= 4(-3a^2)\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$$

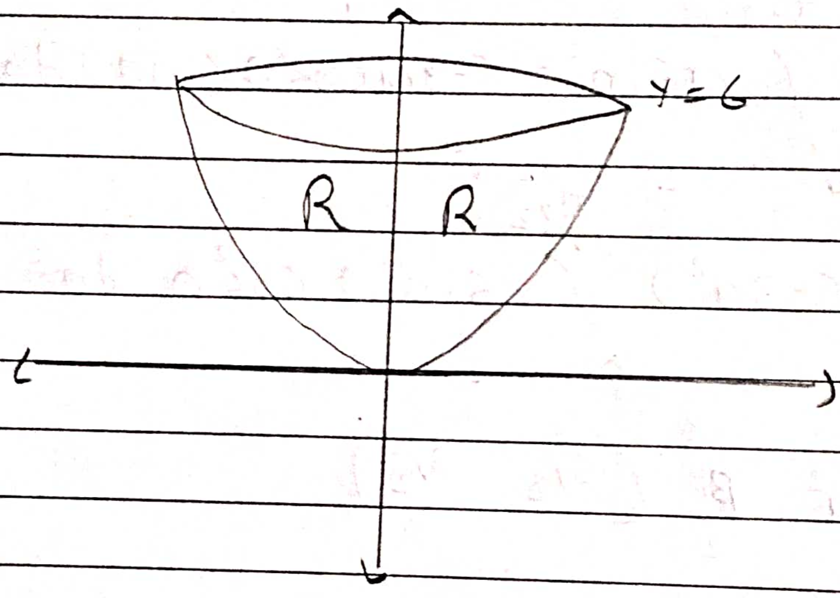
$$A = -12a^2\,\frac{\beta}{2}\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$A = -6a^2 \times \frac{3/2 \times 1/2 \times \sqrt{\pi} \times 1/2 \times \sqrt{\pi}}{3\times 2\times 1}$$

$$A = \frac{3\pi a^2}{8}$$

* Task-2 : Volume of solid of Revolution

1 Find the volume of the solid obtained by rataing the region bounded $y = x^2$, $y = 6$ and $x = 0$ about y-axis.



$$\text{Volume} = \int_0^6 \pi x^2 \, dy$$

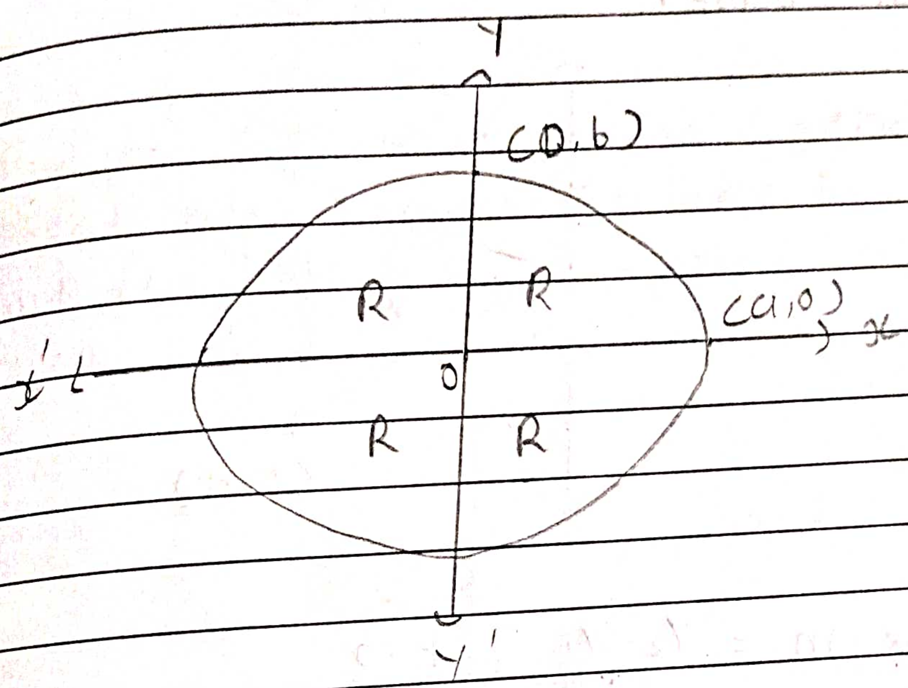
$$= \int_0^6 y^2 \pi \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^6$$

$$V = 18\pi$$

SMVS
Page No.
Date: / /

2 Find the volume of the solid obtained by rotating the ellipe $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x - axis.



$$V = 2\int_0^a \pi y^2 \, dx$$

$$= 2\pi \int_0^a b^2 - \frac{b^2 x^2}{a^2} \, dx$$

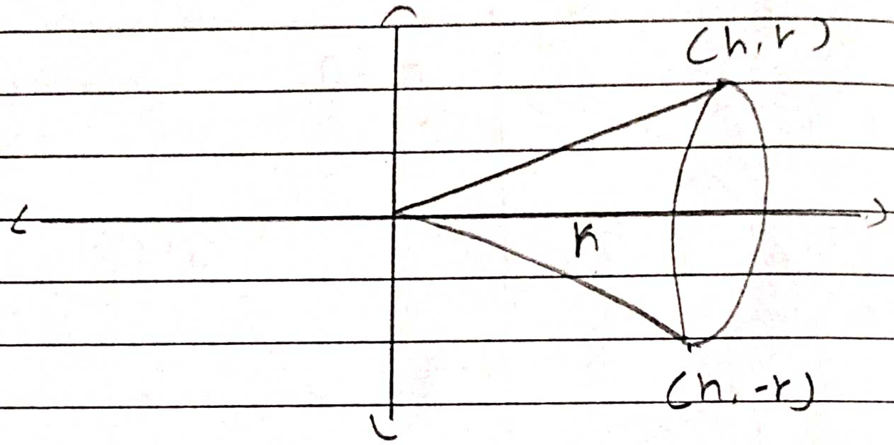
$$= 2\pi b^2 \int_0^a 1 - \frac{x^2}{a^2} \, dx$$

$$= 2\pi b^2 \left[x - \frac{x^3}{3a^2} \right]_0^a$$

$$= 2\pi b \left[a - \frac{a^3}{3a^2} \right]$$

$$V = \frac{4b^2\pi a}{3}$$

3 Find the volume of cone of height h and base r.



=> Slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{r-0}{h-0}$

$$\therefore x_2 - x_1 = \frac{h}{r}(y_2 - y_1)$$

$$\therefore \frac{xr}{h} = y$$

$$\text{Volume} = \int_0^h \pi y^2 \, dx$$

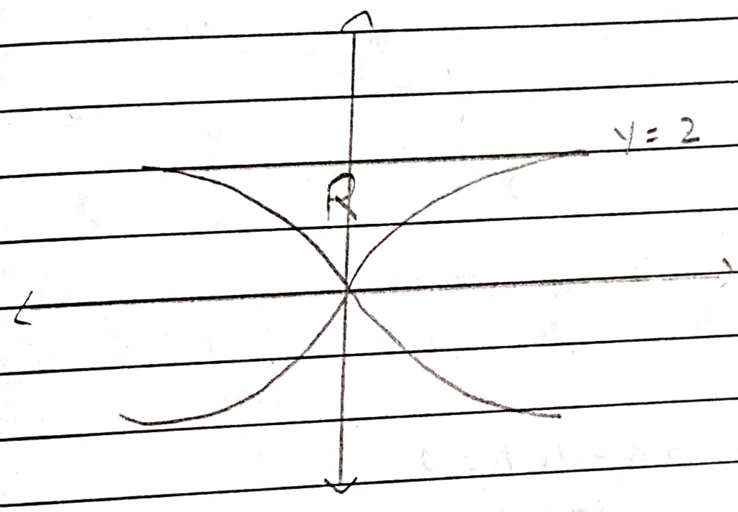
$$= \pi \int_0^h \left(\frac{xr}{h}\right)^2 dx$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$V = \frac{r^2\pi}{h}\left[\frac{h^3}{3}\right]$$

$$V = \frac{\pi r^2 h}{3}$$

4 Find the volume of the solid generated by rotating the region bounded by $y=\sqrt{x}$ and the line $y=2$, $x=0$ about the line $x=0$.



Here $y=\sqrt{x}$

$\therefore x^2 = y^4$

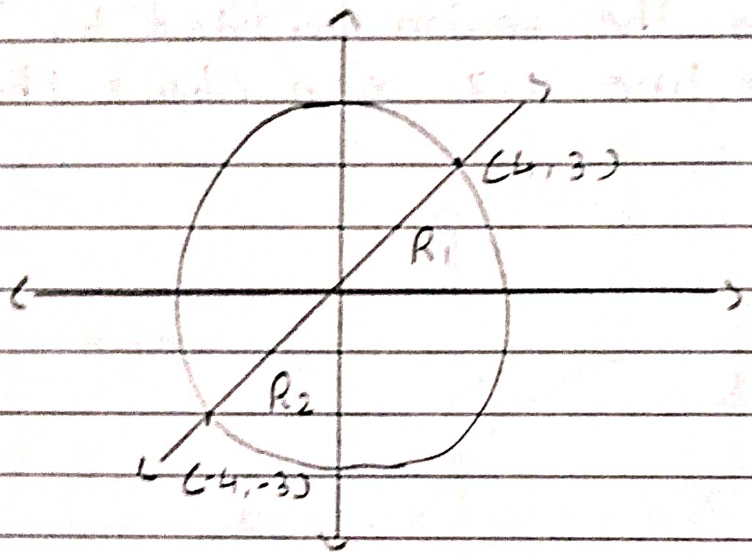
$$\text{Volume} = \int_0^2 \pi x^2 \, dy$$

$$= \pi \int_0^2 y^4 \, dy$$

$$= \pi \left[\frac{y^5}{5}\right]_0^2$$

$$V = \frac{32\pi}{5}$$

5 Find the volume generated by rotating about x-axis the area bounded by the curve $x^2+y^2=25$, $3x-4y=0$ and $y=0$.



$\Rightarrow$ Here $3x-4y=0$

$$\therefore x = \frac{4y}{3}$$

$$\therefore \cancel{x^2+(}$$

$\Rightarrow$ $3x-4y=0$ $\therefore y = \dfrac{3x}{4}$

$$\therefore x^2 + \left(\frac{3x}{4}\right)^2 = 25$$

$$\therefore x = \pm 4$$

Page No.
Date : / /

$\Rightarrow x = +4$ For $y = \frac{3(4)}{4} = 3$

$x = -4$ For $y = \frac{3(-4)}{4} = -3$

$\Rightarrow$ Volume $= 2\left[\int_0^4 y_1^2 \pi \, dx + \int_4^5 y_2^2 \pi \, dx\right]$

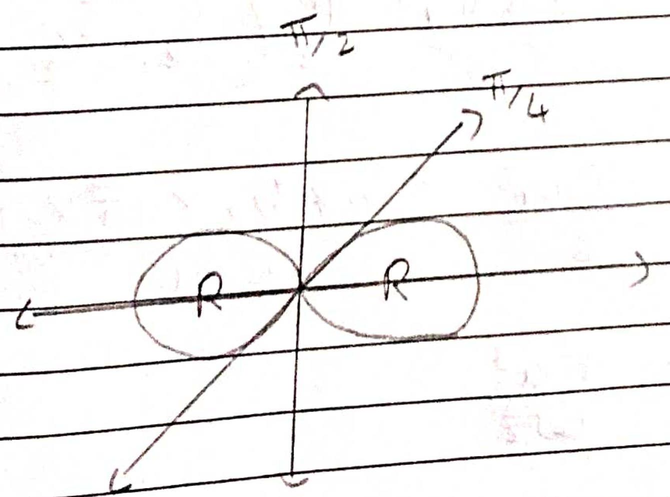
$= 2\left[\int_0^4 \pi\left(\frac{3x}{4}\right)^2 dx + \int_4^5 \pi(25 - x^2)\, dx\right]$

$= 2\pi\left(\left[\frac{9}{16}\left[\frac{x^3}{3}\right]_0^4\right] + \left[25x - \frac{x^3}{3}\right]_4^5\right)$

$= 2\pi\left(\frac{9}{16}\left[\frac{64}{3}\right] + \left(25(5) - \frac{5^3}{3}\right) - \left(25(4) - \frac{4^3}{3}\right)\right)$

$= \frac{100\pi}{3}$

6 Find the volume of the solid generated by revolving the curve $r^2 = a^2 \cos 2\theta$



$$\text{Volume} = 2\int_0^{\pi/4} \frac{2\pi}{3} r^3 \, d\theta \cdot \cos\theta$$

$$= \frac{4\pi}{3}\int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \cdot d\theta$$

Suppose $\sqrt{2}\sin\theta = \sin\phi$

$\therefore \sqrt{2}\cos\theta \, d\theta = \cos\phi \cdot d\phi$

$\theta \to \frac{\pi}{4} \to \phi = \pi/2$

$\theta \to 0 \to \phi = 0$

$$\text{Volume} = \frac{4\pi a^3}{3}\int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \cdot \frac{\cos\phi}{\sqrt{2}} \, d\phi$$

$$= \frac{\overset{2\sqrt{2}}{\cancel{4}}\pi a^3}{3}\int_0^{\pi/2} \cos^3\phi \cdot \sin^0\phi \, d\phi$$

$$= \frac{2\sqrt{2}\pi a^3}{3} \frac{\beta}{2}\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$V = \frac{2\sqrt{2}\pi a^2}{6} \frac{\times 3/2 \times 1/2 \times \sqrt{\pi} \times \sqrt{\pi}}{2 \times 1}$$

$$V = \frac{\pi^2 a^3}{4\sqrt{2}}$$